MATH1061 — Week 3 Notes

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Lecture 7 — Direct Proof, Counterexample

Pre-work: Video 009 (Direct Proofs and Counterexamples). First lecture of the number theory and methods of proof topic — see proof-techniques[1] for the reference definitions, proof templates and the catalogue of common proof errors.

Learning goals

- How to write a direct proof.
- How to disprove a statement by exhibiting a counterexample.

Definitions to know

Proofs and disproofs need the precise definitions of even, odd, prime, composite, rational, divides, … Each of the six below is a **biconditional**, because a proof reads a definition in both directions: unfolding it in the hypothesis, and folding it back up in the conclusion.

n is an **even** integer	$\leftrightarrow$	$n = 2k$ for some integer k
n is an **odd** integer	$\leftrightarrow$	$n = 2k + 1$ for some integer k
n is a **prime** number	$\leftrightarrow$	$n \in \mathbb{Z}$, $n > 1$, and $\forall r, s \in \mathbb{Z}^+$, if $n = rs$ then ($r = 1$ and $s = n$) or ($r = n$ and $s = 1$)

[1] courses/math1061/proof-techniques.pdf

n is a **composite** number	$\leftrightarrow$	$n \in \mathbb{Z}$, $n > 1$, and $\exists r, s \in \mathbb{Z}^+$ such that $n = rs$ and $1 < r < n$ and $1 < s < n$
r is a **rational** number	$\leftrightarrow$	$\exists a, b \in \mathbb{Z}$ such that $r = \frac{a}{b}$ and $b \neq 0$
$d \mid n$	$\leftrightarrow$	$d \neq 0$ and $n = dk$ for some integer k

Two points about these definitions:

- **"Odd" has a second, equally valid form.** $n = 2k+1$ for some integer k and $n = 2k-1$ for some integer k are equivalent: they define the same set, and you may finish a proof in whichever form drops out of the algebra. The proof that $\frac{x}{2}$ is odd, below, is where it pays off: it lands on the $2k-1$ form.
- **The $b \neq 0$ in the definition of rational is its own requirement**, separate from $r = \frac{a}{b}$. A "rational" claim is two claims: that the fraction exists, *and* that its denominator is non-zero. Dropping the second half is the standard way to lose the mark — see rational-and-irrational-numbers[2].

How much detail to give in a proof. Demonstrate facts **using the definitions**. That is the whole answer, and it is why the definitions come first.

Two facts, and where "not prime" bites

> **Fact**: the sum, difference, and product of integers is an integer.
>
> **Fact**: every integer is either even or odd.

The first is what licenses lines like "since $k \in \mathbb{Z}$, we have $3k + 2 \in \mathbb{Z}$" — cheap, but it has to be *said*, because the definition of odd demands an integer witness and nothing else supplies it.

What "not prime" means. For $n \in \mathbb{Z}^{>1}$, "n is not prime" means "n is composite". For $n \in \mathbb{Z}$ generally, "n is not prime" means "composite **or** $n \leq 1$". Primality and compositeness only carve up the integers from 2 upwards; 1, 0 and the negatives are in neither class.

The asymmetry between the two panels is the point. The left one is a genuine two-case split you may argue by; the right one is not, and a proof that reasons "n is not prime, so n is composite" is only sound once $n \geq 2$ has been established. The proof about $n^2 + 5n + 6$ below turns on exactly that.

[2] courses/math1061/rational-and-irrational-numbers.pdf

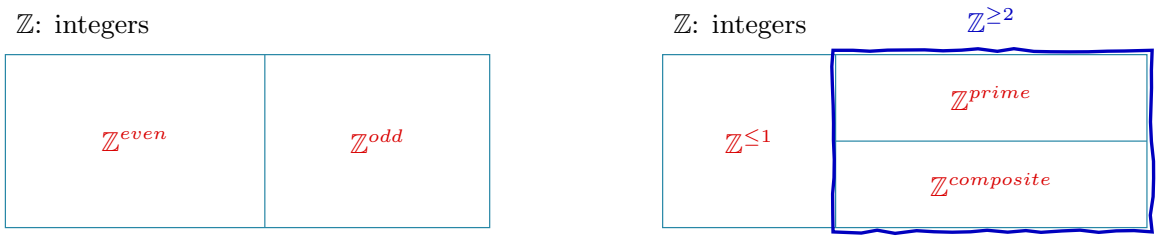


Figure 1: Two ways of splitting $\mathbb{Z}$. Left: even and odd exhaust the integers, which is what makes 'every integer is either even or odd' usable as a case split. Right: prime and composite do *not* — they only partition $\mathbb{Z}^{\geq 2}$, and everything from 1 downwards falls in $\mathbb{Z}^{\leq 1}$. The heavy blue box round the two right-hand cells marks $\mathbb{Z}^{\geq 2}$, the same restriction as $\mathbb{Z}^{>1}$.

Proof or disproof of $\forall x \in D, P(x) \to Q(x)$

Every method in this topic attacks one row of the truth table for $p \to q$: the only row that makes it false is p true, q false.

Why a direct proof supposes $P(x)$ is true. $P(x) \to Q(x)$ would still be true if $P(x)$ is false, so that case needs no argument. We just need to rule out the case that $P(x)$ is true and $Q(x)$ is false — so we suppose $P(x)$ is true and prove that in this case $Q(x)$ must also be true. The other three rows of the table are true for free, so there is nothing to do in them.

The mirror image is what makes a counterexample work: because the statement fails as soon as *one* x occupies that row, exhibiting one such x is a complete disproof. See conditional-statements[3] for the truth table itself.

Worked: which values give a counterexample? (pre-class Q1)

Which values for m and n give a counterexample disproving: *for all integers m and n, if $2m+n$ is odd, then m and n are both odd*?

	Check	Verdict
a. $m = 3, n = 5$	$2(3) + 5 = 11$ odd, but m, n are both odd — so q is **true**	not a counterexample
b. $m = 3, n = 6$	$2(3) + 6 = 12$ even — so p is **false**	not a counterexample

[3] courses/math1061/conditional-statements.pdf

	Check	Verdict
c. $m = 4, n = 5$	$2(4) + 5 = 13$ odd, and m, n are **not** both odd (at least one is even)	**counterexample**
d. $m = 4, n = 6$	$2(4) + 6 = 14$ even — so p is **false**	not a counterexample

For a counterexample to $p \to q$ we require p **true** and q **false**. Options (b) and (d) fail for the opposite reason to (a): they never even engage the statement, because a conditional with a false hypothesis is true and so proves nothing either way.

The wording for (c) matters: not "m and n are both even", but "**not both** m, n are odd — at least one is even". Negating "both odd" gives "at least one is even", not "both even". That is the De Morgan step this question is testing. See 2026-08-06-negation-and-multiple-quantifiers[4].

Example proof

$\forall n \in \mathbb{Z}$, if n is odd then $3n + 2$ is odd.

The shape to memorise:

1. **State the hypothesis** in the skeleton's words — `Let n  and suppose n is odd.`
2. **Unfold the definition**, naming a witness: $n = 2k + 1$ for some $k \in \mathbb{Z}$.
3. **Do the algebra until the target definition is visible** — $6k + 5$ is not obviously odd; $2(3k + 2) + 1$ is, because it matches the shape $2(\text{integer}) + 1$ exactly.
4. **Confirm the new witness is an integer**, by the Fact above.
5. **State the conclusion** in the target's own words.

Step 4 is the one most often skipped. $2(3k + 2) + 1$ only certifies "odd" if $3k + 2$ is an integer, and saying so costs one line.

Worked: if $x + 6 = 4y$ then $\frac{x}{2}$ is odd

For any integer x, if $x + 6 = 4y$ for some integer y, then $\frac{x}{2}$ is an odd integer.

> *Proof.* Let $x \in \mathbb{Z}$ and suppose $x + 6 = 4y$ for some $y \in \mathbb{Z}$. Now $x = 4y - 6$, so $x = 2(2y - 3)$. Hence $\dfrac{x}{2} = \dfrac{2(2y-3)}{2} = 2y - 3 = 2(y - 1) - 1$. Thus, because $y \in \mathbb{Z}$, we have $2y - 3 \in \mathbb{Z}$ so $\frac{x}{2} \in \mathbb{Z}$; and $y - 1 \in \mathbb{Z}$, so by definition $\frac{x}{2}$ is odd. $\square$

Three things this proof drills:

[4] courses/math1061/2026-08-06-negation-and-multiple-quantifiers.pdf

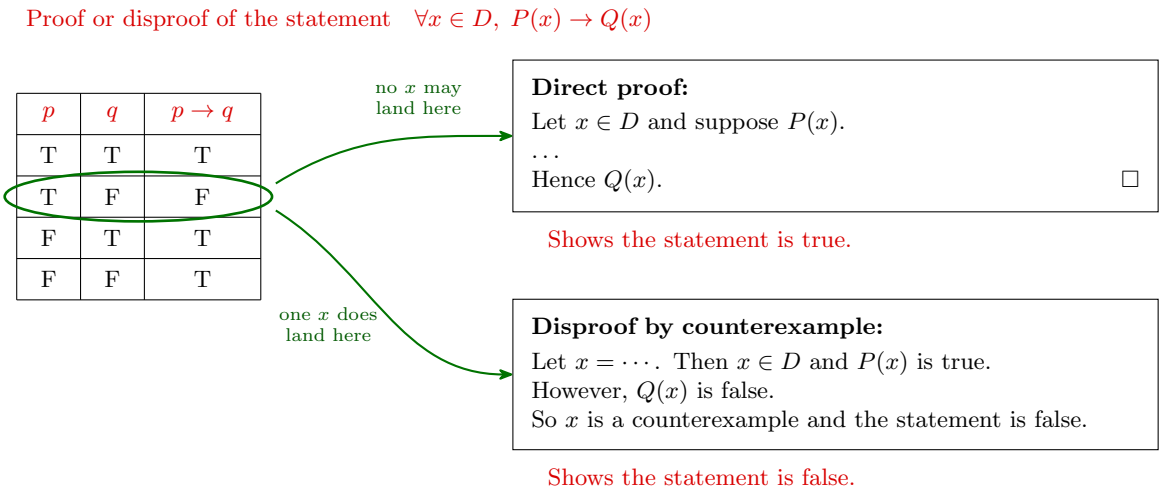


Figure 2: The truth table for $p \to q$ beside the skeletons of a direct proof and a disproof by counterexample. The green ring marks the T-F-F row: it is the only row that can make $P(x) \to Q(x)$ false, so it is the only row either method has to deal with. A direct proof shows *no* $x \in D$ can land there; a counterexample exhibits *one* x that does. The two green arrows join that row to the two skeletons.

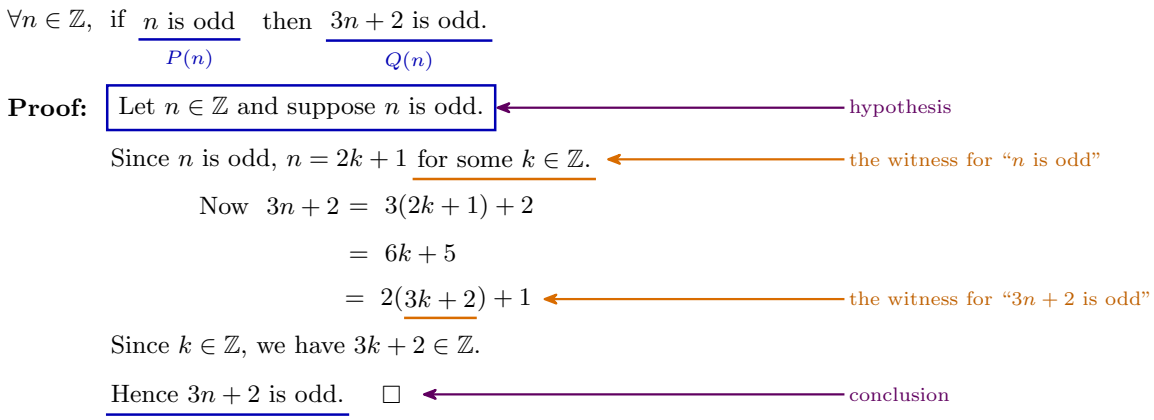


Figure 3: The anatomy of a direct proof. The opening line is boxed, the purple margin labels mark the *hypothesis* and the *conclusion*, and the orange underlines and margin captions mark the two integer witnesses. The blue underlines on the statement pick out its two halves, $P(n)$ and $Q(n)$, which ties the proof back to the skeleton above.

- **The hypothesis is itself existential.** "$x + 6 = 4y$ for some integer y" hands you a y; the opening line accepts it by name and the rest of the proof uses it.
- **The conclusion is a conjunction, and both halves need discharging.** $\frac{x}{2}$ must be shown to be an *integer* (via $2y - 3 \in \mathbb{Z}$) and to have the *odd* form. Neither half implies the other.
- **Either form of "odd" finishes it.** The proof ends on $2(y-1)-1$, and $2(y-2)+1$ does just as well. This is the $n = 2k - 1$ equivalence from the definitions, cashed in.

Worked: prove or disprove

The primes begin $2, 3, 5, 7, 11, ...$

(a) There are distinct integers m and n such that $\frac{1}{m} + \frac{1}{n}$ is an integer.

True. Take $m = 1$ and $n = -1$. Then $\frac{1}{m} + \frac{1}{n} = 1 + (-1) = 0 \in \mathbb{Z}$.

The statement is existential, so one witness *is* the proof — the opposite of the situation with $n^2 + 5n + 6$ below. The word **distinct** matters: it rules out the easy $m = n = 1$, so the witness uses a negative integer instead.

(b) $\forall n \in \mathbb{Z}^{\geq 2}$, n is composite or $n + 1$ is composite.

False. Take $n = 2$: then $n = 2$ and $n + 1 = 3$, which are both prime.

This one is a universal statement that is *not* a conditional — the predicate is a disjunction. So the counterexample has to falsify the disjunction outright, i.e. make **both** disjuncts false, which here means finding two consecutive primes. $n = 2$ is the *only* value that works, because for $n \geq 3$ one of n, $n + 1$ is even and bigger than 2, hence composite.

(c) For each integer $n \geq 2$, the product of the first n prime numbers minus 1 is prime.

False. Try $n = 2, 3, 4, ...$ in turn. Take $n = 4$:

$$2 \cdot 3 \cdot 5 \cdot 7 - 1 = 209 = 11 \cdot 19 \quad \text{— composite.}$$

The two values before it: $n = 2$ gives $2 \cdot 3 - 1 = 5$, prime; $n = 3$ gives $2 \cdot 3 \cdot 5 - 1 = 29$, prime. So the first two cases both hold and the claim only breaks at the third. That is why "arguing from examples" is on the list of common proof errors in proof-techniques[5]. A pattern surviving two checks is not evidence of anything.

[5] courses/math1061/proof-techniques.pdf

Worked: disprove by proving the negation

Show that $\exists n \in \mathbb{Z}^+$ such that n^2+5n+6 is prime is false, by proving the negation.

The statement is existential, so a counterexample won't do — one bad n says nothing about whether some *other* n works. Negating it turns it into a universal, and that has to be proved outright:

> Negation: $\forall n \in \mathbb{Z}^+$, n^2+5n+6 is composite.
>
> *Proof of the negation.* Let $n \in \mathbb{Z}^+$. Now $n^2+5n+6 = (n+2)(n+3)$. Since $n \geq 1$ we have $n+2 > 1$ and $n+3 > 1$. Thus n^2+5n+6 is composite. $\square$

The original statement is false (**F**) and its negation is true (**T**). The step from "not prime" to "composite" needs a justification: because $n \geq 1$ we have $n^2+5n+6 \geq 2$, so "not prime" is "composite". The value sits in $\mathbb{Z}^{\geq 2}$, the region of Figure 1 where the two really do coincide. Without that check the negation of "prime" would also have to admit values ≤ 1.

One small gap to close: the proof gives only the two "> 1" halves, but the definition of composite asks for factors with $1 < r < n$ **and** $1 < s < n$. Here each factor is also strictly less than the product, since the other factor exceeds 1 — so $n+2 < (n+2)(n+3)$ and $n+3 < (n+2)(n+3)$, and the definition is met in full.

The general move — **to disprove $\exists x \in D$ such that $P(x)$, prove $\forall x \in D, \sim P(x)$** — is the quantifier negation rule from 2026-08-06-negation-and-multiple-quantifiers[6] used as a proof strategy. It generalises: a universal falls to one counterexample, an existential needs a universal proof.

See also

- proof-techniques[7] · quantified-statements[8] · conditional-statements[9]
- rational-and-irrational-numbers[10] · divisibility-and-factorisation[11] — the two definitions from the table that get their own notes
- 2026-08-13-proof-by-contradiction[12] — Lecture 8, which adds a third skeleton to the two above
- 2026-08-13-proof-by-contraposition[13] — Lecture 9, which adds the fourth

[6] courses/math1061/2026-08-06-negation-and-multiple-quantifiers.pdf
[7] courses/math1061/proof-techniques.pdf
[8] courses/math1061/quantified-statements.pdf
[9] courses/math1061/conditional-statements.pdf
[10] courses/math1061/rational-and-irrational-numbers.pdf
[11] courses/math1061/divisibility-and-factorisation.pdf
[12] courses/math1061/2026-08-13-proof-by-contradiction.pdf
[13] courses/math1061/2026-08-13-proof-by-contraposition.pdf

- math1061-diagram-notation[14] — the house conventions the figures above follow
- practice-problems[15] — §9 Direct Proofs and Counterexamples, with full worked solutions

Lecture 8 — Proof by Contradiction

Pre-work: Video 010 (Proof by Contradiction). Adds a third method to the direct proof and counterexample of 2026-08-10-direct-proof-and-counterexample[16] — see proof-techniques[17] for all four side by side.

Learning goals

- How to write a proof by contradiction.

Recall

The negations this method depends on (see logical-equivalence-laws[18] and quantified-statements[19]):

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$
$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$
$$\sim (p \to q) \equiv p \wedge \sim q$$
$$\sim (\exists x \in D \text{ such that } P(x)) \equiv \forall x \in D, \sim P(x)$$
$$\sim (\forall x \in D, P(x)) \equiv \exists x \in D \text{ such that } \sim P(x)$$

[14] courses/math1061/math1061-diagram-notation.pdf
[15] courses/math1061/practice-problems.pdf
[16] courses/math1061/2026-08-10-direct-proof-and-counterexample.pdf
[17] courses/math1061/proof-techniques.pdf
[18] courses/math1061/logical-equivalence-laws.pdf
[19] courses/math1061/quantified-statements.pdf

The definitions every proof below unfolds

The definitions are written as **biconditionals**, because a proof uses them in both directions — read left-to-right to unpack a hypothesis, right-to-left to establish a conclusion.

- n is an **even** integer $\leftrightarrow$ $n = 2k$ for some integer k
- n is an **odd** integer $\leftrightarrow$ $n = 2k+1$ for some integer k $\leftrightarrow$ $n = 2k-1$ for some integer k
- n is a **prime** number $\leftrightarrow$ $n \in \mathbb{Z}$, $n > 1$, and $\forall r, s \in \mathbb{Z}^+$, if $n = rs$ then ($r = 1$ and $s = n$) or ($r = n$ and $s = 1$)
- n is a **composite** number $\leftrightarrow$ $n \in \mathbb{Z}$, $n > 1$, and $\exists r, s \in \mathbb{Z}^+$ such that $n = rs$ and $1 < r < n$ and $1 < s < n$
- r is a **rational** number $\leftrightarrow$ $\exists a, b \in \mathbb{Z}$ such that $r = \frac{a}{b}$ and $b \neq 0$
- $d \mid n \leftrightarrow d \neq 0$ and $n = dk$ for some integer k

Both forms of "odd" are usable: $2k+1$ and $2k-1$ describe the same set of integers, and some calculations come out tidier with one than the other. The same definitions are tabulated in proof-techniques[20]; the rational one is unpacked in rational-and-irrational-numbers[21] and the divisibility one in divisibility-and-factorisation[22].

Fact: the sum, difference and product of integers is an integer.

Warning: don't assume something that seems like an "obvious fact" until you have proven it, or can say which lecture or pre-work video it was proved in.

When to use contradiction

Three questions share one answer: whether proof by contradiction can be used for every proof, whether it only works for universal statements or just works best for them, and which situations suit it over a direct proof. Proof by contradiction is the **most versatile** method, but it requires more set-up, so sometimes a direct proof is easier.

Proof by counterexample and proof by contradiction do opposite jobs. Counterexample is a **disproof**; contradiction is a **proof**.

The three methods so far

There are three proof skeletons so far (Figure 4); the fourth arrives next lecture.

For a proof by contradiction we begin by **assuming that the statement is false** — equivalently, that the negation of the statement is true. The negation of $\forall x \in D, P(x) \to Q(x)$ is

[20] courses/math1061/proof-techniques.pdf
[21] courses/math1061/rational-and-irrational-numbers.pdf
[22] courses/math1061/divisibility-and-factorisation.pdf

Proofs of the statement $\forall x \in D,\ P(x) \to Q(x)$

Direct proof:
Let $x \in D$ and suppose $P(x)$.
...
Hence $Q(x)$. □

Shows the statement is true.

Disproof by counterexample:
Let $x = \cdots$. Then $x \in D$ and $P(x)$ is true.
However, $Q(x)$ is false.
So x is a counterexample and the statement is false.

Shows the statement is false.

Proof by contradiction:
Suppose the negation of $\forall x \in D, P(x) \to Q(x)$ is true.
That is, suppose $\exists x \in D$ such that $P(x)$ is true but $Q(x)$ is false.
...
Hence something that is obviously wrong, like $P(x)$ is false, or $Q(x)$ is true, or $1 = 0$.
This is a contradiction, and we conclude that $\forall x \in D, P(x) \to Q(x)$ is true. □

(⇒⇐)

Shows the statement is true.

Figure 4: Three proof skeletons for the same statement, each with its verdict in red underneath. Only the contradiction box starts by assuming the statement false; the other two start from the statement's own hypothesis. The orange (⇒⇐) in the margin marks the point where the contradiction lands.

$$\exists x \in D \text{ such that } P(x) \wedge \sim Q(x)$$

and we derive from it something that cannot hold — $P(x)$ false, or $Q(x)$ true, or $1 = 0$.

Note what the two moves in that negation are: the $\forall$ becomes an $\exists$ (so you are handed *one* specific x to work with, not an arbitrary one), and $\sim (P \to Q) \equiv P \wedge \sim Q$ turns the implication into a conjunction (so you get to assume the hypothesis **and** the failure of the conclusion at once). Both extra assumptions are why the method is the most versatile one, and why it costs more set-up.

The contradiction mark. (⇒⇐) — two arrows meeting head-on — is written inline at the point where the contradiction lands. Every proof by contradiction below uses it. It is a marker, not a substitute for saying in words *what* contradicts *what*.

Worked: how to start a proof by contradiction

Claim: $\forall n \in \mathbb{Z}$, if n is odd then $3n + 2$ is odd. An appropriate way to start a proof by contradiction is:

Proof: Suppose that n is an odd integer and that $3n + 2$ is even.

which means **exactly the same thing** as

> Suppose $\exists n \in \mathbb{Z}$ such that n is odd but $3n + 2$ is even.

The second phrasing is the one the negation rule actually produces, so it is the safer one to write while you are learning; the first is the same sentence with the existential left implicit.

Worked: what type of proof starts this way?

Claim: for all integers m and n, if mn is even, then m is even or n is even. Consider the first sentence of a proof:

> *Proof:* Suppose that m and n are integers, mn is even and both m and n are odd.

This is the start of a **proof by contradiction**, not a direct proof and not a disproof by counterexample. It assumes the hypothesis *and* the negation of the conclusion ("m even or n even" negates to "m odd and n odd" by De Morgan).

Worked: the same statement, both ways

$\forall n \in \mathbb{Z}$**, if** n **is odd then** $3n + 2$ **is odd.**

It can be proved both ways (Figure 5).

Direct proof

> Let $n \in \mathbb{Z}$ and suppose n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$ (by definition). Now
>
> $$3n + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1.$$
>
> Since $k \in \mathbb{Z}$, we know $3k + 2 \in \mathbb{Z}$, and hence $3n + 2$ is odd (by definition). □

Proof by contradiction

> Suppose $\exists n \in \mathbb{Z}$ such that n is odd and $3n + 2$ is **even**. Then (since n is odd), $n = 2k + 1$ for some $k \in \mathbb{Z}$. Now
>
> $$3n + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1.$$
>
> Since $k \in \mathbb{Z}$, we know $3k + 2 \in \mathbb{Z}$, hence $3n + 2$ is **odd** ($\Rightarrow\Leftarrow$) — contradicting the assumption that it is even. Therefore the original statement is true. □

The body of the two proofs is identical; contradiction just wraps it in an extra assumption and an extra closing line. That's the "more set-up" cost — and here the direct proof is the better choice.

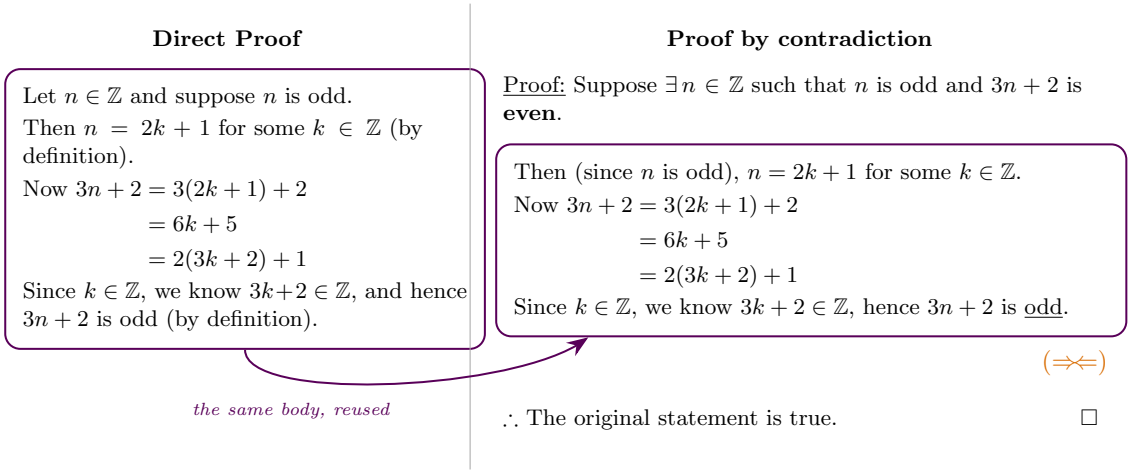


Figure 5: The same statement proved both ways — direct proof left, proof by contradiction right. The purple outline marks the body the two proofs share, and the arrow carries it across: it is character-for-character the same algebra. Contradiction only adds an assumption line on top and the (⇒⇐) plus conclusion at the bottom. That extra wrapper is the 'more set-up' that contradiction costs.

Worked: where contradiction earns its keep

$\forall m, n \in \mathbb{Z}$**, if** mn **is even, then** m **is even or** n **is even.**

Direct proof — suppose $m, n \in \mathbb{Z}$ and mn is even. We can write $mn = 2k$ for some $k \in \mathbb{Z}$ … and now there's no way forward: knowing $mn = 2k$ tells you nothing about m and n individually.

Proof by contradiction

> Suppose $\exists m, n \in \mathbb{Z}$ such that mn is even and both m and n are odd. Since m is odd, $m = 2k + 1$ for some $k \in \mathbb{Z}$. Since n is odd, $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$ — **use a new variable**; m and n need not be equal. Now
>
> $$mn = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1.$$
>
> Since $k, \ell \in \mathbb{Z}$, $2k\ell + k + \ell \in \mathbb{Z}$, so mn is odd. It is impossible for mn to be both even and odd (⇒⇐). Therefore the original statement is true. □

Reusing k for both m and n is one of the standard proof errors listed in proof-techniques[23]. Note also where the contradiction lands here: not on a stray arithmetic absurdity but on the *hypothesis itself* — the assumed-even mn turns out to be odd. That is one of the three landing points listed above ("$P(x)$ is false").

[23] courses/math1061/proof-techniques.pdf

Worked: if $3n^3 - 2$ is odd then n is odd

Prove by contradiction: $\forall n \in \mathbb{Z}$, if $3n^3 - 2$ is odd, then n is odd.

> *Proof.* Suppose the statement is false. That means $\exists n \in \mathbb{Z}$ such that $3n^3 - 2$ is odd and n is even. Since n is even, $n = 2a$ for some $a \in \mathbb{Z}$. Now
>
> $$3n^3 - 2 = 3(2a)^3 - 2 = 3 \cdot 8a^3 - 2 = 24a^3 - 2 = 2(12a^3 - 1).$$
>
> Since $a \in \mathbb{Z}$, $12a^3 - 1 \in \mathbb{Z}$, so $3n^3 - 2$ is even ($\Rightarrow\Leftarrow$) — a contradiction. Therefore the original statement is true. $\square$

The negation is worth reading slowly: "if $3n^3 - 2$ is odd then n is odd" negates to "$3n^3 - 2$ is odd **and** n is even". The even hypothesis is the one that gives you something to substitute, so that is where the work starts.

Worked: without loss of generality

$\forall m, n \in \mathbb{Z}$, if $m + n$ is even, then m and n are both even or m and n are both odd.

> *Proof.* Suppose the statement is false. Thus $\exists m, n \in \mathbb{Z}$ such that $m + n$ is even but exactly one of m, n is even (and the other is odd).
>
> **Without loss of generality** (WLOG), assume m is even and n is odd. (The two cases are (1) m even, n odd and (2) m odd, n even; since $m + n = n + m$, the second is the first with the names swapped, so proving one proves both.)
>
> Thus $m = 2a$ for some $a \in \mathbb{Z}$, and $n = 2b + 1$ for some $b \in \mathbb{Z}$. Now
>
> $$m + n = (2a) + (2b + 1) = 2(a + b) + 1.$$
>
> We know $a + b \in \mathbb{Z}$ because $a, b \in \mathbb{Z}$, so $m + n$ is odd. This contradicts the assumption that $m + n$ is even ($\Rightarrow\Leftarrow$). Therefore the original statement is true. $\square$

"Without loss of generality" is only legitimate when the omitted case really is the stated one relabelled — say why, as above. The phrase abbreviates to **WLOG**.

See also

- proof-techniques[24] · quantified-statements[25] · logical-equivalence-laws[26]
- 2026-08-13-proof-by-contraposition[27] — Lecture 9
- practice-problems[28] — §10 Proof by Contradiction, with full worked solutions

[24] courses/math1061/proof-techniques.pdf
[25] courses/math1061/quantified-statements.pdf
[26] courses/math1061/logical-equivalence-laws.pdf
[27] courses/math1061/2026-08-13-proof-by-contraposition.pdf
[28] courses/math1061/practice-problems.pdf

Lecture 9 — Proof by Contraposition, Rational Numbers

Pre-work: Video 011 (Proof by Contraposition), Video 012 (Rational Numbers). Completes the set of four methods — see proof-techniques[29] for the full comparison, and rational-and-irrational-numbers[30] for the $\mathbb{Q}$ / $\overline{\mathbb{Q}}$ definitions and closure results.

Learning goals

1. Understand the definitions of rational and irrational numbers.
2. Gain fluency in proof techniques (direct proof, proof by contradiction and proof by contraposition).

Recall: $r \in \mathbb{Q} \iff r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$; and $r \in \overline{\mathbb{Q}} \iff r \notin \mathbb{Q}$, where $\overline{\mathbb{Q}} = \mathbb{R} \setminus \mathbb{Q}$. So $\frac{3}{4} \in \mathbb{Q}$, while $\sqrt{2} \in \overline{\mathbb{Q}}$ and $\pi \in \overline{\mathbb{Q}}$. The barred $\overline{\mathbb{Q}}$ is this course's symbol for the irrationals — see rational-and-irrational-numbers[31] for the picture.

The four methods

Method	Shape	Shows
Direct proof	Let $x \in D$ and suppose $P(x)$. ... Hence $Q(x)$.	true
Disproof by counterexample	Let $x = \cdots$. Then $x \in D$ and $P(x)$ is true. However $Q(x)$ is false.	false
Proof by contradiction	Suppose $\exists x \in D$ such that $P(x)$ is true but $Q(x)$ is false. ... Hence something obviously wrong.	true
Proof by contraposition	We prove $\forall x \in D, \sim Q(x) \to \sim P(x)$. Let $x \in D$ and suppose $\sim Q(x)$. ... Hence $\sim P(x)$.	true

This is the table from 2026-08-13-proof-by-contradiction[32], grown by one row. Contraposition is valid because

[29] courses/math1061/proof-techniques.pdf
[30] courses/math1061/rational-and-irrational-numbers.pdf
[31] courses/math1061/rational-and-irrational-numbers.pdf
[32] courses/math1061/2026-08-13-proof-by-contradiction.pdf

$$p \to q \equiv \sim q \to \sim p$$

(see conditional-statements[33]). Contraposition does *not* assume the statement is false. It is a **direct proof of a different statement** — one that happens to be logically equivalent to the one you were asked about. That is the practical difference from proof by contradiction, which assumes $P(x) \wedge \sim Q(x)$ and hunts for an absurdity.

Choosing a method

What is the relationship between proof by contradiction and proof by contraposition? As above: contraposition is a direct proof of an equivalent statement, while contradiction assumes $P(x) \wedge \sim Q(x)$ and hunts for an absurdity.

How do we decide if we should use a direct proof, proof by contradiction or proof by contraposition? Empirically rather than by a rule: for each statement, ask what type of proof will work, and see which methods move and which ones stall. The verdicts for the statements proved below are collected in Figure 6.

The pattern that emerges: **a method works when it hands you something you can substitute for.** A direct proof needs the hypothesis to be a positive, unfoldable statement; contraposition needs the *negated conclusion* to be one; contradiction gets both at once, which is why its column never gets a cross.

The last lemma from Video 12, by contraposition

Lemma: the sum of any rational number and any irrational number is irrational.

Symbolically, either of:

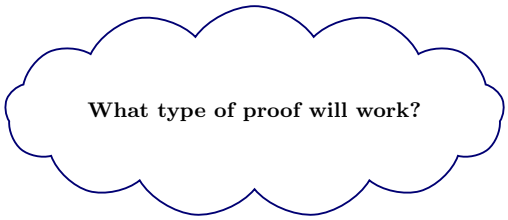
$$\forall r, s \in \mathbb{R}, \text{ if } r \in \mathbb{Q} \text{ and } s \in \overline{\mathbb{Q}} \text{ then } r + s \in \overline{\mathbb{Q}}$$

$$\forall r \in \mathbb{Q} \text{ and } \forall s \in \mathbb{R}, \text{ if } s \in \overline{\mathbb{Q}} \text{ then } r + s \in \overline{\mathbb{Q}}$$

The second is the form to work with, because it puts the whole conditional in terms of one unknown s: the antecedent is $s \in \overline{\mathbb{Q}}$ and the consequent is $r + s \in \overline{\mathbb{Q}}$, so the contrapositive is a clean swap.

- **Direct proof**: suppose $r \in \mathbb{Q}$ and $s \in \overline{\mathbb{Q}}$. Now we are stuck — $s \in \overline{\mathbb{Q}}$ gives nothing to substitute.
- **Proof by contradiction**: suppose $\exists r \in \mathbb{Q}$ and $s \in \mathbb{R}$ such that $s \in \overline{\mathbb{Q}}$ and $r + s \in \mathbb{Q}$. (Worked in Video 012.)

[33] courses/math1061/conditional-statements.pdf



Statement	**Direct proof**	**Proof by contradiction**	**Proof by contraposition**
Lemma (p3): if $r \in \mathbb{Q}$ and $s \in \overline{\mathbb{Q}}$ then $r+s \in \overline{\mathbb{Q}}$	✗ stuck	✓ in the video	✓
Activity 1: if $r^2 \in \overline{\mathbb{Q}}$ then $r \in \overline{\mathbb{Q}}$	✗	✓	✓
Activity 2: $\forall r \in \mathbb{Q},\ (2r^2-3r+1) \in \mathbb{Q}$	✓ (Best)	✓	✗ stuck.
Activity 3: if mn is odd then m and n are both odd	• (not as easy)	✓	✓
Activity 4: $\sqrt{2} \in \overline{\mathbb{Q}}$	✗	•	✗ ?

Figure 6: Which type of proof works for each statement proved below. A tick is a method that works and a cross is one that stalls, with a short remark in red where there is one; a plain bullet is a method given neither. Reading down the columns: contradiction is the only method never ruled out, which is what 'most versatile' means in practice — but it is also never the one labelled 'Best'.

- **Proof by contraposition**:

  The contrapositive is: $\forall r \in \mathbb{Q}$ and $\forall s \in \mathbb{R}$, if $r + s \in \mathbb{Q}$ then $s \in \mathbb{Q}$.

  Suppose $r \in \mathbb{Q}$, $s \in \mathbb{R}$ and $r + s \in \mathbb{Q}$. Since $r \in \mathbb{Q}$, $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$. Since $r + s \in \mathbb{Q}$, $r + s = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ where $d \neq 0$ — a **new pair of variables**, since nothing says the two fractions are related. Now

  $$s = (r + s) - r = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}.$$

  Since $a, b, c, d \in \mathbb{Z}$ we know $bc - ad \in \mathbb{Z}$ and $bd \in \mathbb{Z}$. Also $bd \neq 0$ since $b \neq 0$ and $d \neq 0$. Thus $s \in \mathbb{Q}$. $\square$

The move that unlocks it is writing $s = (r + s) - r$ — expressing the unknown in terms of the two things you were handed. The closing three checks (integer numerator, integer denominator, non-zero denominator) are what the definition of $\mathbb{Q}$ actually demands, and every rational-number proof in this course ends with them.

Worked: if r^2 is irrational then r is irrational

Prove that $\forall r \in \mathbb{R}$, if r^2 is irrational, then r is irrational.

Which type of proof will work: contradiction and contraposition both work; a direct proof does not.

> *Proof.* We prove the contrapositive: $\forall r \in \mathbb{R}$, if $r \in \mathbb{Q}$ then $r^2 \in \mathbb{Q}$. Suppose $r \in \mathbb{R}$ where $r \in \mathbb{Q}$. Then $r = \frac{a}{b}$ for some integers a, b with $b \neq 0$. Now $r^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$. Since $a, b \in \mathbb{Z}$ we have $a^2, b^2 \in \mathbb{Z}$; also $b^2 \neq 0$ since $b \neq 0$. So $r^2 \in \mathbb{Q}$. $\square$

Both negations do real work here. "r^2 is irrational" is a statement about what r^2 is *not*, and so is "r is irrational" — the contrapositive turns both into positive membership claims, and a positive claim is one you can substitute a fraction into.

Worked: $2r^2 - 3r + 1$ is rational when r is

Prove that $\forall r \in \mathbb{Q}, (2r^2 - 3r + 1) \in \mathbb{Q}$.

Which type of proof will work: a direct proof works and is the **best**; contradiction also works; contraposition gets stuck.

Proof. Let $r \in \mathbb{Q}$. Then $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$. Now

$$2r^2 - 3r + 1 = 2\left(\frac{a}{b}\right)^2 - 3\left(\frac{a}{b}\right) + 1 = \frac{2a^2}{b^2} - \frac{3ab}{b^2} + \frac{b^2}{b^2} = \frac{2a^2 - 3ab + b^2}{b^2}.$$

Since $a, b \in \mathbb{Z}$ we have $2a^2 - 3ab + b^2 \in \mathbb{Z}$ and $b^2 \in \mathbb{Z}$. Also $b^2 \neq 0$ since $b \neq 0$. So $2r^2 - 3r + 1 \in \mathbb{Q}$. □

In the middle step everything is put over the common denominator b^2 *before* the numerator is inspected, so that the final expression is visibly one integer over one non-zero integer.

Contraposition stalls here because the negation of "$2r^2 - 3r + 1 \in \mathbb{Q}$" — that it's irrational — gives nothing to substitute. The direction of the useful information decides the method.

Worked: if mn is odd then m and n are both odd

Prove by contraposition: $\forall m, n \in \mathbb{Z}$, if mn is odd, then m and n are both odd.

Which type of proof will work: contradiction and contraposition both work; a direct proof is not as easy.

Label the two halves of the statement p and q first, and only then write the contrapositive underneath: forming a contrapositive is a mechanical two-step operation on those labelled parts, not a fresh piece of thinking (Figure 7).

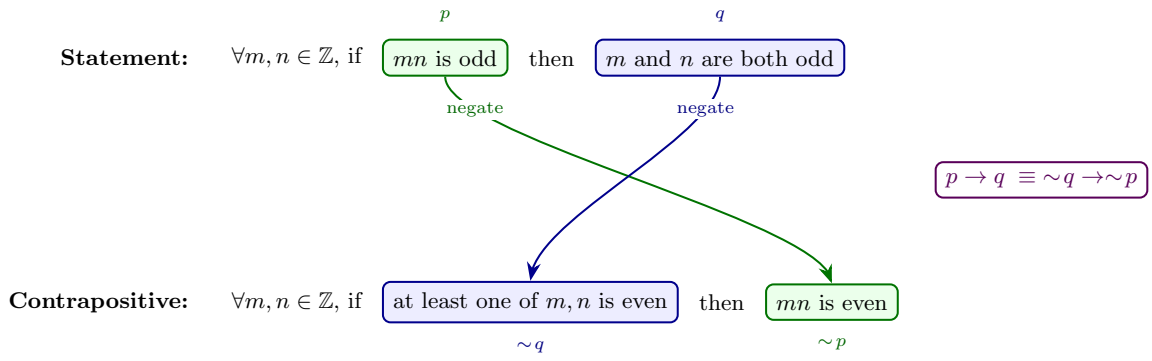


Figure 7: Forming the contrapositive of 'if mn is odd then m and n are both odd', with the parts labelled p and q. Two things happen at once and both are needed: each half is negated, and the two halves swap sides. The crossing arrows are what 'swap' looks like. The negation of q is the De Morgan step — 'm and n are both odd' becomes 'at least one of m, n is even' — which is where this kind of proof is most often got wrong.

Proof. We prove the contrapositive: $\forall m, n \in \mathbb{Z}$, if at least one of m, n is even then mn is even. Without loss of generality, assume m is even. So $m = 2k$ for some $k \in \mathbb{Z}$. Now $mn = (2k)n = 2(kn)$. Since $k, n \in \mathbb{Z}$, $kn \in \mathbb{Z}$, so mn is even. □

"m and n are both odd" negates to "m is even **or** n is even" by De Morgan — hence "at least one of". The WLOG is what lets one line cover both disjuncts: the roles of m and n in mn are interchangeable, so the case "n is even" is the case above with the two names swapped.

The contrapositive needs very little algebra — three lines, and n is never unfolded at all. That is the payoff, and why the direct proof is "not as easy".

Worked: $\sqrt{2}$ is irrational

Which type of proof will work: **contradiction**. A direct proof does not work. Contraposition does not apply, because $\sqrt{2} \in \overline{\mathbb{Q}}$ is not a conditional at all, so there is no antecedent and consequent to swap.

Two ingredients:

- **Fact**: every rational number can be written as a fraction in lowest terms (numerator and denominator have no common factors).
- **Lemma (Video 11)**: for all integers n, if n^2 is even then n is even.

  Proof. Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$, and we assume $\frac{a}{b}$ is in **lowest terms**.

  Now $(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$, so $2 = \frac{a^2}{b^2}$ and hence $a^2 = 2b^2$; thus a^2 is even. By the lemma, a **is even**, so $a = 2\ell$ for some $\ell \in \mathbb{Z}$.

  Then $2b^2 = a^2 = (2\ell)^2 = 4\ell^2$, so $b^2 = 2\ell^2$ and b^2 is even. By the lemma again, b **is even**.

  This contradicts the assumption that $\frac{a}{b}$ is in lowest terms — a and b share the factor 2 ($\Rightarrow\Leftarrow$).

  Therefore $\sqrt{2} \in \overline{\mathbb{Q}}$. $\square$

The "in lowest terms" assumption is what the contradiction lands on; without it the argument has nothing to contradict. The whole chain — including the two separate applications of the lemma — is drawn out in rational-and-irrational-numbers[34].

See also

- proof-techniques[35] · rational-and-irrational-numbers[36] · conditional-statements[37]
- 2026-08-13-proof-by-contradiction[38] — Lecture 8

[34] courses/math1061/rational-and-irrational-numbers.pdf
[35] courses/math1061/proof-techniques.pdf
[36] courses/math1061/rational-and-irrational-numbers.pdf
[37] courses/math1061/conditional-statements.pdf
[38] courses/math1061/2026-08-13-proof-by-contradiction.pdf

- 2026-08-17-divisibility[39] — Lecture 10
- practice-problems[40] — §11 Proof by Contraposition, with full worked solutions
- practice-problems[41] — §12 Rational Numbers, with full worked solutions

Reference material

Conditional & Biconditional Statements

Conditional ($\rightarrow$)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Only one row makes a conditional false: the row where the hypothesis p is true and the conclusion q is false. Everything else is true, including both rows where p is false.

$p \rightarrow q$ is true when p is false **because that is how it is defined.** There is no deeper intuition to recover; $\rightarrow$ is a truth-functional connective fixed by the table above, and arguing with the bottom two rows is arguing with a definition. The standard name for those two rows is **vacuous truth**. It does real work later: it is why $\emptyset \subseteq A$ for every set A (see sets-and-set-notation[42]).

The picture behind the table is containment:

Disjunctive form

By definition,

$$p \rightarrow q \equiv \sim p \vee q$$

The right-hand side is the **disjunctive form** of the conditional, and it is the single most useful fact in the topic: the laws of logical equivalence (see logical-equivalence-laws[43]) are stated only for $\sim$, $\wedge$ and $\vee$, so they say nothing about $\rightarrow$ directly. To use the laws on a conditional:

[39] courses/math1061/2026-08-17-divisibility.pdf
[40] courses/math1061/practice-problems.pdf
[41] courses/math1061/practice-problems.pdf
[42] courses/math1061/sets-and-set-notation.pdf
[43] courses/math1061/logical-equivalence-laws.pdf

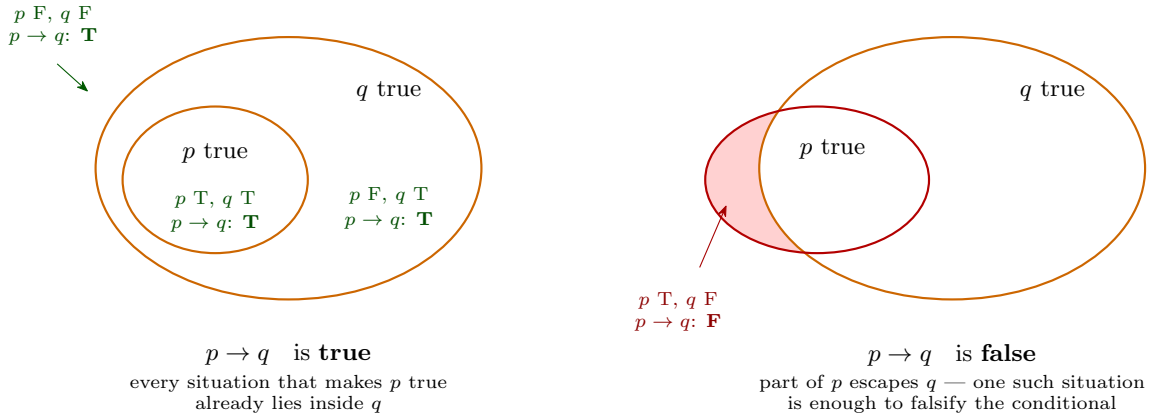


Figure 8: Why the truth table has the shape it does. Left: p → q asserts that the situations making p true all lie inside the situations making q true, so every region carries T. Right: the conditional fails exactly when part of p escapes q, which is the single F row of the table.

Replace $p \to q$ with $\sim p \vee q$ and then use the laws.

Every worked proof about conditionals in this course starts and (usually) ends with that substitution.

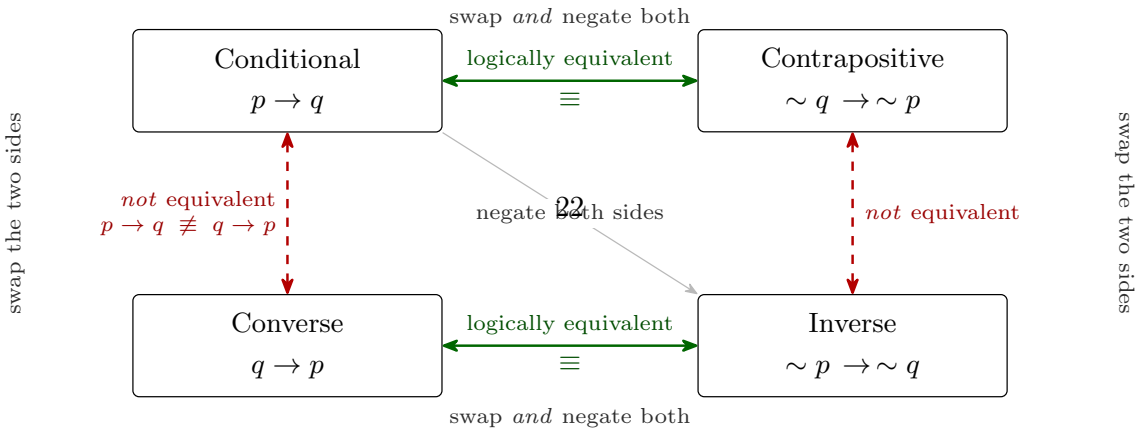
$p \to q$ can be phrased: "p implies q", "if p then q", "q if p", "p is sufficient for q", "p only if q", "q is necessary for p".

Warning: $p \to q \not\equiv q \to p$.

Contrapositive, converse, inverse

Name	Form	Equivalent to
Conditional	$p \to q$	Contrapositive
Contrapositive	$\sim q \to \sim p$	Conditional
Converse	$q \to p$	Inverse
Inverse	$\sim p \to \sim q$	Converse

The conditional and its **contrapositive** are always logically equivalent. The **converse** and **inverse** are always logically equivalent to each other — but *not* to the original conditional.



Both equivalences fall straight out of the disjunctive form. For the first: $p \to q \equiv\sim p \vee q$ and $\sim q \to\sim p \equiv\sim (\sim q) \vee \sim p \equiv q \vee \sim p$, which is the same disjunction commuted. The converse/inverse pair is the same computation with p and q interchanged.

Necessary and sufficient conditions

- p is a **necessary condition** for q means "if $\sim p$ then $\sim q$", or equivalently "if q then p", or "q **only if** p". In symbols, $q \to p$.
- p is a **sufficient condition** for q means "if p then q", or equivalently "q **if** p". In symbols, $p \to q$.

The two phrasings of "necessary" are contrapositives of each other, which is why they say the same thing: $\sim p \to\sim q$ and $q \to p$.

The reliable reading is: **"if" introduces the sufficient condition; "only if" introduces the necessary one.**

Worked: necessary but not sufficient

Being a non-negative integer is necessary for a number to be the square of an integer, but it is not sufficient.

- *Necessary*: if a number is the square of an integer, then it is non-negative. So -9 cannot be the square of an integer — failing the condition rules the number out.
- *Not sufficient*: 7 is a non-negative integer but it is not the square of an integer. Meeting the condition does not settle it.

Worked: sufficient but not necessary

Being an integer that is divisible by 4 is sufficient for being an even integer, but it is not necessary.

- *Sufficient*: if an integer is divisible by 4, then it is even. 8 is divisible by 4, so we know 8 is even without checking anything else.
- *Not necessary*: 10 is an even integer but 10 is not divisible by 4. The condition can fail while the conclusion still holds.

The two shapes are exactly opposite in how they are refuted: a claimed **sufficient** condition is killed by something satisfying the condition but not the conclusion; a claimed **necessary** condition is killed by something satisfying the conclusion but not the condition.

Biconditional ($\leftrightarrow$)

$$a \leftrightarrow b \equiv (a \to b) \wedge (b \to a)$$

A biconditional is the two one-way conditionals conjoined, and each of the standard English phrasings splits along the same seam:

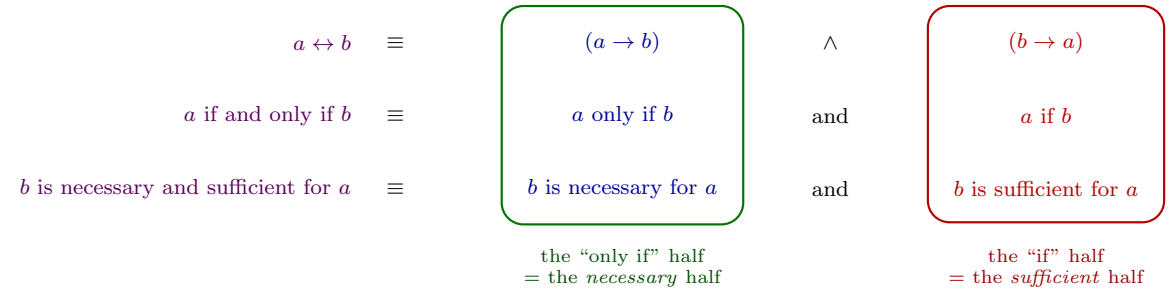


Figure 10: The biconditional decomposed three ways, with each half in its own ringed column. Reading down the left column: a → b, 'a only if b', 'b is necessary for a' are three names for the same half. Reading down the right: b → a, 'a if b', 'b is sufficient for a'. The whole necessary/sufficient vocabulary is just this correspondence — 'only if' is the necessary half, 'if' is the sufficient half.

Negating a conditional

$$\sim (p \to q) \equiv \sim (\sim p \vee q) \equiv p \wedge \sim q$$

by the definition of →, then De Morgan's law, then the double negative law.

The negation of a conditional is not a conditional — it's an *and*. "If it rains, the match is cancelled" is negated by "it rains **and** the match is not cancelled", not by "if it rains, the match is not cancelled".

The classic error is to answer $\sim p \to \sim q$ (that's the *inverse*) or $q \to p$ (the *converse*). Neither is the negation, and neither is even equivalent to the original — see Figure 9 above.

Worked: $\sim (p \to q) \not\equiv \sim p \to \sim q$**.** Let p be **true** and q be **true**. Then $p \to q$ is true, so $\sim (p \to q)$ is **false**. But $\sim p \to \sim q$ is F → F, which is **true**. The two differ on this row, so they are not equivalent.

Sanity check on the shapes: $\sim (p \to q)$ is a conjunction and is true on exactly **one** of the four rows; $\sim p \to \sim q$ is a conditional and is true on **three**. They could hardly be equivalent.

Order of operations

1. $\sim$
2. $\wedge, \vee$ (equal)
3. $\rightarrow, \leftrightarrow$ (equal)

Within a tier, work left to right if no parentheses are given. So $p \rightarrow \sim q \vee r$ means $p \rightarrow ((\sim q) \vee r)$ — but parenthesise explicitly rather than relying on the tie-breaking rule.

Notation, not to be blurred: the biconditional $\leftrightarrow$ is a connective used to **combine** statements (like $+$, $-$, $\times$, $\div$ with numbers). Logical equivalence $\equiv$ is used when **comparing** two statements (like $=$ with numbers). $a \leftrightarrow b$ is a statement; $a \equiv b$ is a claim *about* two statements.

See also

- logical-connectives[44]
- logical-equivalence-laws[45]
- sets-and-set-notation[46] — vacuous truth doing real work, in $\emptyset \subseteq A$
- 2026-07-30-conditional-statements[47] — Lecture 3
- practice-problems[48] — §3 Conditional Statements, with full worked solutions

Divisibility & Unique Factorisation

Definition

$$d \mid n \iff d \neq 0 \text{ and } n = dk \text{ for some integer } k$$

Both n and d must be integers, and $d \neq 0$ — this side condition is part of the definition, not an afterthought. Equivalently, $d \mid n \iff \frac{n}{d} \in \mathbb{Z}$.

All of these say the same thing:

- d divides n
- d is a divisor of n
- d is a factor of n
- n is a multiple of d

[44] courses/math1061/logical-connectives.pdf
[45] courses/math1061/logical-equivalence-laws.pdf
[46] courses/math1061/sets-and-set-notation.pdf
[47] courses/math1061/2026-07-30-conditional-statements.pdf
[48] courses/math1061/practice-problems.pdf

$d \nmid n$ means d does not divide n. Note that $d \mid n$ is a **statement**, not a number: $2 \mid 6$ is *true*, it is not "2 divided by 6" and it is not 3. Likewise $10 \nmid 5$ is true, since $\frac{5}{10} \notin \mathbb{Z}$. Being a statement is what makes $2 \mid x$ a **predicate** — it acquires a truth value once x is fixed — so it is exactly the kind of thing a quantifier can be put in front of.

Divisibility inside a quantified statement

Two quantified divisibility statements, both **false**, both settled by a single counterexample — one counterexample is enough to make a $\forall$ statement false, though no single example could ever make one true (see quantified-statements[49] on that asymmetry):

Statement	Verdict	Counterexample
$\forall x \in \mathbb{Z}$, if $2 \mid x$ then $4 \mid x$	**False**	$x = 2$: $2 \mid 2$ is true but $4 \nmid 2$
$\forall x \in \mathbb{Z}$, if 6 is divisible by x, then $x = 2$	**False**	$x = 3$: $3 \in \mathbb{Z}$ and $3 \mid 6$, but $3 \neq 2$

A counterexample to a universal conditional has to make the **hypothesis true and the conclusion false** — that is what $\sim (p \to q) \equiv p \wedge \sim q$ says. An x that fails the hypothesis proves nothing.

The second statement becomes true once the conclusion lists every divisor:

$$\forall x \in \mathbb{Z},\ \text{if 6 is divisible by } x,\ \text{then } x = \pm 1 \text{ or } \pm 2 \text{ or } \pm 3 \text{ or } \pm 6.$$

The $\pm$ is not optional. The reason is in the definition: it asks only for $n = dk$ with $k \in \mathbb{Z}$, and k may be negative, so divisors over $\mathbb{Z}$ always come in $\pm$ pairs — if $n = dk$ then $n = (-d)(-k)$, so $d \mid n$ exactly when $(-d) \mid n$. A divisor list that quietly restricts itself to $\mathbb{Z}^+$ is answering a question about a different domain.

Proving and disproving divisibility statements

The pattern for a direct proof: unfold $d \mid n$ into $n = dk$, do algebra, then re-fold into the definition by exhibiting the integer witness.

$\forall a, b, c \in \mathbb{Z}$, if $c \mid a$ then $c \mid ab$.

> Let $a, b, c \in \mathbb{Z}$ and suppose $c \mid a$. Then $\exists k \in \mathbb{Z}$ such that $a = ck$. Multiplying both sides by b: $ab = ckb = c(kb)$. Since $k, b \in \mathbb{Z}$ we have $kb \in \mathbb{Z}$. Thus $c \mid ab$. $\square$

$\forall m \in \mathbb{Z}$, $6m(2m + 10m^2)$ is divisible by 4.

[49] courses/math1061/quantified-statements.pdf

Let $m \in \mathbb{Z}$. Then $6m(2m+10m^2) = 2(3m)\cdot 2(m+5m^2) = 4(3m(m+5m^2))$. Since $m \in \mathbb{Z}$, $3m(m+5m^2) \in \mathbb{Z}$. Thus $4 \mid 6m(2m+10m^2)$. $\square$

$\forall a, b, c \in \mathbb{Z}$**, if** $a \mid (b+c)$ **then** $a \mid b$ **or** $a \mid c$ — **false**. Counterexample $a = 3, b = 4, c = 2$: $a \mid (b+c)$ since $3 \mid 6$, but $3 \nmid 4$ and $3 \nmid 2$.

$\forall a, b, m \in \mathbb{Z}$**, if** $m \mid (a+b)$ **then** $m \mid (a-b)$ — **false**. A counterexample needs the hypothesis true and the conclusion false:

	$m \mid (a+b)$	$m \mid (a-b)$	conditional
$m = 5, a = 20, b = 10$	$5 \mid 30$ T	$5 \mid 10$ T	T
$m = 5, a = 21, b = 13$	$5 \mid 34$ F	$5 \mid 8$ F	T
$m = 5, a = 27, b = 7$	$5 \mid 34$ F	$5 \mid 20$ T	T
$m = 5, a = 19, b = -9$	$5 \mid 10$ T	$5 \mid 28$ F	**F**
$m = 5, a = 23, b = 12$	$5 \mid 35$ T	$5 \mid 11$ F	**F**
$m = 5, a = -10, b = -3$	$5 \mid -13$ F	$5 \mid -7$ F	T

Only the two rows with hypothesis T and conclusion F are counterexamples.

$\forall m, n, q \in \mathbb{Z}$**, if** $m \mid n$ **and** $n \mid q$ **then** $m^2 \mid nq$ — **true**.

Let m, n, q be non-zero integers with $m \mid n$ and $n \mid q$. By definition there exist $a, b \in \mathbb{Z}$ with $n = ma$ and $q = nb$. Then

$$nq = (ma)(nb) = ma \cdot (mab) = m^2(a^2 b).$$

Since $a^2 b \in \mathbb{Z}$, we conclude $m^2 \mid nq$. $\square$

If k **is even and** m **is odd, then** $(k+2)^2 - (m-3)^2$ **is divisible by 4.**

Let $k = 2a$ and $m = 2b+1$ for some $a, b \in \mathbb{Z}$. Then $(k+2)^2 = (2a+2)^2 = 4(a+1)^2$ and $(m-3)^2 = (2b-2)^2 = 4(b-1)^2$, so

$$(k+2)^2 - (m-3)^2 = 4\big((a+1)^2 - (b-1)^2\big),$$

and $(a+1)^2 - (b-1)^2 \in \mathbb{Z}$. $\square$

$\forall c, d, e \in \mathbb{Z}$**, if** $c \mid d$ **and** $c \nmid e$**, then** $c \nmid (d+e)$**.**

Proof by contradiction. Suppose there exist integers c, d, e with $c \mid d$, $c \nmid e$ and $c \mid (d+e)$. Then $d = cx$ for some $x \in \mathbb{Z}$, and $d + e = cy$ for some $y \in \mathbb{Z}$. Hence $cx + e = cy$, so $e = c(y-x)$ with $y - x \in \mathbb{Z}$ — so $c \mid e$, a contradiction. $\square$

Every integer $n > 1$ is a product of primes

Proof by contradiction. Suppose the theorem is false. Then there exists an integer $n > 1$ that is not a product of primes. **Choose the smallest such** n — such a smallest one exists by the well-ordering principle. Choosing the smallest is what lets us assume every integer strictly between 1 and n *is* a product of primes.

Every integer $n > 1$ is either prime or composite (not both).

Case 1: n is prime. Then n is a product of primes (namely itself) — contradiction.

Case 2: n is composite. Then $n = rs$ for positive integers r, s with $r \neq 1$ and $s \neq 1$, which forces $1 < r < n$ and $1 < s < n$. Because n was chosen smallest, both r and s are products of primes, say $r = p_1 p_2 \cdots p_k$ and $s = q_1 q_2 \cdots q_\ell$. Therefore $n = rs = p_1 \cdots p_k q_1 \cdots q_\ell$ is a product of primes too.

Either way n is a product of primes, contradicting our choice of n. □

Unique Factorisation Theorem

Provided in the exam. This theorem and the Quotient-Remainder Theorem are both stated on the *MATH1061/MATH7861 Examination Formula Page*, so they can be cited rather than restated from memory — see also logical-equivalence-laws[50] for the rest of what that page carries.

Given any integer $n > 1$, the **standard factored form** of n is

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

where $k \in \mathbb{Z}^+$, the p_i are primes, the e_i are positive integers, and $p_1 < p_2 < \cdots < p_k$.

Both the primes and their exponents are unique — this is what makes the factored form a canonical fingerprint of an integer, and it's the engine behind divisor counting, gcd and lcm (see gcd-lcm-and-euclidean-algorithm[51]).

Counting divisors from the factorisation

Any positive divisor of $n = p_1^{e_1} \cdots p_k^{e_k}$ has the form $p_1^{a_1} \cdots p_k^{a_k}$ with $0 \leq a_i \leq e_i$, so there are

$$(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$$

positive divisors.

[50] courses/math1061/logical-equivalence-laws.pdf
[51] courses/math1061/gcd-lcm-and-euclidean-algorithm.pdf

$6975 = 3^2 \cdot 5^2 \cdot 31$, found by trial division: $6975 = 3 \cdot 2325 = 3 \cdot 3 \cdot 775 = 3 \cdot 3 \cdot 5 \cdot 155 = 3^2 \cdot 5^2 \cdot 31$. A divisor has 0, 1 or 2 threes; 0, 1 or 2 fives; and 0 or 1 thirty-ones — so $3 \cdot 3 \cdot 2 = 18$ positive divisors.

$27720 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$**.**

$1400 = 2^3 \cdot 5^2 \cdot 7$, worked through:

- $1400 = 2(700) = 2 \cdot 2(350) = 2 \cdot 2(35)(10) = 2^2(5 \cdot 7)(2 \cdot 5) = 2^3 \cdot 5^2 \cdot 7^1$ — note the primes are written in increasing order.
- $1400^2 = (2^3 \cdot 5^2 \cdot 7)^2 = 2^6 \cdot 5^4 \cdot 7^2$ — raising to a power **multiplies** the exponents.
- $1400^3 = 1400 \cdot 1400^2 = 2^9 \cdot 5^6 \cdot 7^3$ — multiplying **adds** the exponents.
- Number of positive divisors of 1400: $(3 + 1)(2 + 1)(1 + 1) = 24$.
- Divisors of 1400 divisible by 8: all factors of the form $2^3 \cdot 5^* \cdot 7^*$, i.e. $8, 40, 200, 56, 280, 1400$.
- Smallest n with $1400n$ a perfect square: $1400n = t^2$ needs every exponent even. $t^2 = 2^3 \cdot 5^2 \cdot 7^1 \cdot n$, and splitting each power in half (bigger halves on the left) gives $t = 2^2 \cdot 5^1 \cdot 7^1$ against $2^1 \cdot 5^1 \cdot 7^0 \cdot n$, so $n = 2 \cdot 7 = 14$.

Primality testing

Lemma: for every integer $n > 1$, if no integer x in the range $1 < x \leq \sqrt{n}$ divides n, then n is prime.

This is why trial division only has to go as far as $\sqrt{n}$ — for 6975, only up to $\sqrt{6975} = 83.51\ldots$

> *Proof by contraposition.* The contrapositive is: $\forall n \in \mathbb{Z}^{\geq 2}$, if n is not prime, then there exists an integer x with $1 < x \leq \sqrt{n}$ such that $x \mid n$.
>
> Suppose $n \in \mathbb{Z}^{\geq 2}$ and n is not prime. Then n is composite, so by definition $\exists r, s \in \mathbb{Z}^+$ with $n = rs$ and $1 < r < n$ and $1 < s < n$.
>
> *Case 1:* $r = s$. Then $r = s = \sqrt{n}$.
>
> *Case 2:* $r \neq s$. Then one of r, s is less than $\sqrt{n}$ and the other greater — if both were greater we'd have $rs > n$, and if both were less we'd have $rs < n$.
>
> In either case there is an $x \in \{r, s\}$ with $1 < x \leq \sqrt{n}$ and $x \mid n$. $\square$

There are infinitely many primes

Lemma: for any integer a and any prime p, if $p \mid a$ then $p \nmid (a + 1)$.

Proof by contradiction. Let $a \in \mathbb{Z}$, let p be prime, suppose $p \mid a$, and assume for a contradiction that $p \mid (a+1)$. Since $p \mid a$ there exists $k \in \mathbb{Z}$ with $a = pk$; since $p \mid (a+1)$ there exists $\ell \in \mathbb{Z}$ with $a + 1 = p\ell$. So

$$1 = (a+1) - a = p\ell - pk = p(\ell - k),$$

and $\ell - k \in \mathbb{Z}$, so $p \mid 1$. But $p \geq 2$ for any prime, whereas $p \mid 1$ forces $p = \pm 1$ — a contradiction. □

Theorem: there are an infinite number of primes.

Proof by contradiction. Suppose the number of primes is finite. Then some prime p is the largest, and we can list every prime in ascending order: $2, 3, 5, 7, 11, \ldots, p$.

Let $N = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots p) + 1$.

Then $N > 1$, so by the Unique Factorisation Theorem N is a product of primes and hence divisible by some prime q. Because q is prime it must be one of $2, 3, 5, 7, 11, \ldots, p$.

So q divides $2 \cdot 3 \cdot 5 \cdots p$ — that is, $q \mid (N-1)$ — and hence by the lemma $q \nmid ((2 \cdot 3 \cdot 5 \cdots p) + 1) = N$.

So q divides N and q does not divide N: a contradiction. Therefore the original statement is true. □

See also

- proof-techniques[52] · modular-arithmetic[53] · gcd-lcm-and-euclidean-algorithm[54]
- partial-order-relations[55] — "divides" as a partial order on $\mathbb{Z}^+$
- fields[56] — unique factorisation as the reason $\mathbb{Z}_p$ is a field when p is prime
- counting-principles[57] — the rule behind counting divisors, one choice of exponent for each prime
- practice-problems[58] — §13 Divisibility, with full worked solutions

[52] courses/math1061/proof-techniques.pdf
[53] courses/math1061/modular-arithmetic.pdf
[54] courses/math1061/gcd-lcm-and-euclidean-algorithm.pdf
[55] courses/math1061/partial-order-relations.pdf
[56] courses/math1061/fields.pdf
[57] courses/math1061/counting-principles.pdf
[58] courses/math1061/practice-problems.pdf

Logical Equivalence & Laws

Two statement forms are **logically equivalent** ($\equiv$) if they have the same truth value for every combination of truth values of their variables — shown via a truth table, or by chaining the laws below. To show two forms are **not equivalent** ($\not\equiv$), exhibit one combination of truth values where they differ.

$\equiv$ *compares* statements (like $=$ compares numbers); it's not a connective itself. The connective that *combines* two statements into one is the biconditional $\leftrightarrow$ — see conditional-statements[59].

Tautologies and contradictions

> A **tautology t** is a statement that is **always true**. A **contradiction c** is a statement that is **always false**.

Both are written in bold upright lower case on the formula sheet, and both appear as *values* inside the laws — $p \vee \sim p \equiv \mathbf{t}$ produces one, $p \wedge \mathbf{t} \equiv p$ consumes one. A chain of laws that reaches $\mathbf{t}$ or $\mathbf{c}$ has proved the original form is always true or always false, which is usually the point of the exercise.

Laws of logical equivalence

> **Provided in the exam.** These laws are printed on the *MATH1061/MATH7861 Examination Formula Page*, so they don't need memorising. What the exercises ask for is choosing and applying them, and naming the one used at each step. The same page also carries the valid-argument-forms[60] and the set identities of proving-set-properties[61].

Given statement variables p, q, r, a tautology $\mathbf{t}$, and a contradiction $\mathbf{c}$:

Law	Form 1	Form 2
Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
Negation	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
Double negative	$\sim (\sim p) \equiv p$	
Idempotent	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal bound	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$

[59] courses/math1061/conditional-statements.pdf
[60] courses/math1061/valid-argument-forms.pdf
[61] courses/math1061/proving-set-properties.pdf

Law	Form 1	Form 2
De Morgan's	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
Absorption	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of $\mathbf{t}/\mathbf{c}$	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

De Morgan's Laws are especially important for negating "and"/"or" statements — e.g. the negation of "5 is divisible by 2 **and** 6 is divisible by 2" is "5 is **not** divisible by 2 **or** 6 is **not** divisible by 2". Note what changes: the connective flips from *and* to *or*, and **both** halves get negated. Negating only one half, or leaving the connective alone, is the standard error.

How to write a law chain

The layout expected in your own work:

- the **first line is the starting expression**, on its own;
- **every later line begins with** $\equiv$, vertically aligned down the page;
- the **name of the law** used to get there sits in its own **right-hand column** on that line.

Two or more laws may be cited on one line if they act on the same step (e.g. "by De Morgan's law and the double negative law"). Name a law on every line that *changes* the expression. The one kind of line that goes unnamed is a pure re-labelling that asserts nothing new, like the "where $r = (p \wedge \sim q)$" line in the first worked example below.

Treat a compound expression as a single letter

The laws are stated for variables p, q, r, but they hold with **any statement form** substituted in. Spotting that turns a long expression into a short one:

$$(p \wedge \sim q) \wedge \ \sim (p \wedge \sim q)$$

is $r \wedge \sim r$ with $r = (p \wedge \sim q)$, so the negation law applies directly and it is $\mathbf{c}$. This is exactly the move in the first worked example below.

Worked: reaching a contradiction

Show that $(p \wedge \sim q) \wedge (\sim p \vee q) \equiv \mathbf{c}$**.**

$$\begin{aligned}
(p \wedge \sim q) \wedge (\sim p \vee q) &\equiv (p \wedge \sim q) \wedge \ \sim (p \wedge \sim q) && \text{by De Morgan's law and the double negative law} \\
&\equiv r \wedge \sim r \quad \text{where } r = (p \wedge \sim q) \\
&\equiv \mathbf{c} && \text{by the negation law}
\end{aligned}$$

The whole proof is the first line: recognising that $\sim p \vee q$ is $\sim (p \wedge \sim q)$ in disguise. Read De Morgan's backwards — $\sim p \vee q \equiv \sim p \vee \sim (\sim q) \equiv \sim (p \wedge \sim q)$ — and the two conjuncts are a statement and its own negation.

Worked: absorbing a conjunction

Show that $\sim p \vee (p \wedge q) \equiv \sim p \vee q$**.**

$$\begin{aligned}
\sim p \vee (p \wedge q) &\equiv (\sim p \vee p) \wedge (\sim p \vee q) && \text{by the distributive law} \\
&\equiv \mathbf{t} \wedge (\sim p \vee q) && \text{by the negation law} \\
&\equiv \sim p \vee q && \text{by the identity law}
\end{aligned}$$

The pattern *distribute, collapse the tautology or contradiction, drop it with the identity law* is the most common three-step shape in this topic, and it recurs in both routes below.

Worked: two routes to the same answer

Show that $\sim \big((\sim p \wedge q) \vee (\sim p \wedge \sim q)\big) \equiv p$**.**

There is no unique chain. Here are two, and both are complete proofs:

Neither route is "the" answer, and nothing turns on finding the short one. What matters is that every line follows from the one above by a **named** law. The shorter route is found by looking for a **common factor** before doing anything else: both disjuncts inside the brackets contain $\sim p$, so the distributive law pulls it out and the rest collapses.

The same equivalence can also be established with a truth table — see the worked table in 2026-07-30-logical-form-and-equivalence[62], where the column for $\sim \big((\sim p \wedge q) \vee (\sim p \wedge \sim q)\big)$ comes out identical to the column for p.

[62] courses/math1061/2026-07-30-logical-form-and-equivalence.pdf

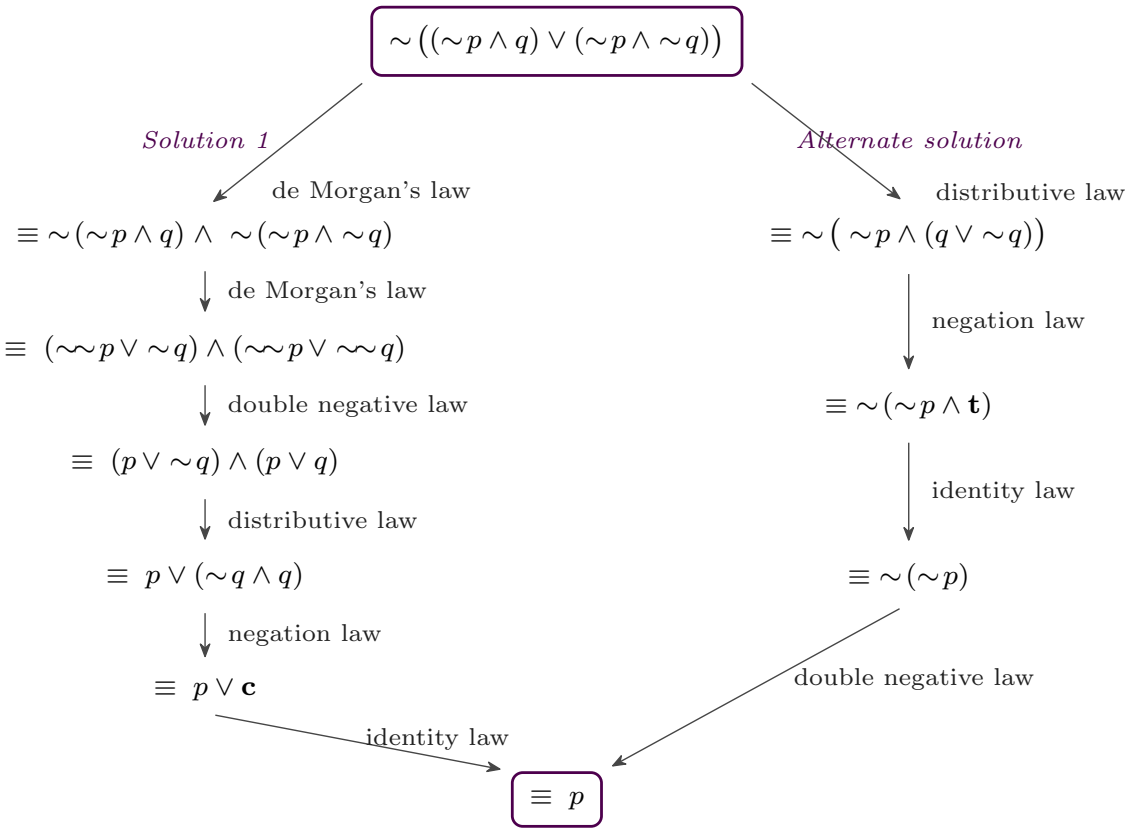


Figure 11: The same equivalence proved two ways. Starting from the same expression and ending at the same p, the left route pushes the negation inward with De Morgan's first and takes six steps; the right route factors out the common ~p with the distributive law first and takes four. Every edge names the law that licenses it — that naming is what the chain is for.

Showing two forms are *not* equivalent

To show $\not\equiv$ you do **not** need a full truth table. Find **one** assignment of truth values on which the two forms disagree, and say what each evaluates to.

Worked. Show that $\sim(p \vee q) \not\equiv \sim p \vee \sim q$.

Take p **false** and q **true**. Then

- LHS: $p \vee q$ is $\mathrm{F} \vee \mathrm{T} = \mathrm{T}$, so $\sim(p \vee q)$ is **false**.
- RHS: $\sim p \vee \sim q$ is $\mathrm{T} \vee \mathrm{F} = \mathrm{T}$, so it is **true**.

One row on which they differ is a complete disproof, so the two forms are **not** logically equivalent.

The correct partner for $\sim(p \vee q)$ is of course $\sim p \wedge \sim q$ — this is precisely the De Morgan trap of changing the sign but not the connective.

See also

- logical-connectives[63]
- conditional-statements[64]
- formula-sheet[65] — what else is provided in the exam, and what is not
- proving-set-properties[66] — the same eleven laws as set identities, and the two set-equality proof methods built on them
- 2026-07-30-logical-form-and-equivalence[67] — Lecture 2, where these laws were introduced and applied
- practice-problems[68] — §2 Logical Equivalence, with full worked solutions

Diagram and Notation Conventions

The house style for every figure, symbol and proof layout in MATH1061 — read off the course's own lecture slides rather than assumed. This is the authority any new diagram for this course is checked against. Where the source does not settle something, that is said so explicitly rather than filled in.

[63] courses/math1061/logical-connectives.pdf
[64] courses/math1061/conditional-statements.pdf
[65] courses/math1061/formula-sheet.pdf
[66] courses/math1061/proving-set-properties.pdf
[67] courses/math1061/2026-07-30-logical-form-and-equivalence.pdf
[68] courses/math1061/practice-problems.pdf

What was read

Everything the course ships in the anchor set, in full: **all 176 slide pages across the 18 lecture decks (L01–L18), plus the 1-page examination formula sheet.** No sampling — the whole corpus was viewed as rendered images so the drawings, ink colours and box shapes were seen, not inferred from extracted text.

Citations below use `L07 p2` = **PDF page 2** of `math1061-lecture-07-…pdf`. Two decks have a page-label offset worth knowing: `L03 p2-p3` carry the printed header *Lecture 2*, and `L17 p9-p10` carry the printed header *Lecture 18*. PDF page position is what is cited.

The course ships one written reference — the *MATH1061/MATH7861 Examination Formula Page*. It is a genuine notation authority (it fixes , , bold **t**/**c**, `A` , , `A - B`, and the numbering of the set identities), but it is **not a drawing guide**: it says nothing about figures, colour, layout or symbol shape. Everything in the *Diagrams* section below is inferred from the decks alone.

Only **18 of the course's 36 lectures** exist in this set (L19–L36 — one-to-one/onto functions, composition, cardinality, relations, equivalence and partial orders, groups, fields, counting, probability, pigeonhole, graph theory, walks, matrix representations, trees — are absent). Conventions for those topics are unestablished; see the last section.

Provenance: two lecturers, one template

The course is split between two lecturers (`L01 p12`, course schedule): **Sara Davies** takes L01–L09 and L18–L26; **Katie Clinch** takes L10–L17 and L27–L36. Both use the same slide frame, but their annotation habits and some box conventions differ. Where they diverge it is recorded under Inconsistencies, not resolved.

Textbook lineage

The recommended text is **Susanna Epp, *Discrete Mathematics with Applications*, 4th or 5th (metric) edition** (`L01 p3`, with the cover image on the slide). It is cited repeatedly by page for statements and exercises — `L14 p7` ("Q27 on page 279 of Epp"), `L14 p8` ("Q31 on page 279"), `L16 p12` ("page 310 (in 4th edition) or page 346 (in 5th edition)"), `L18 p8` ("pages 419-421 (ed. 5) or pages 378-380 (ed. 4)").

Important qualification: no slide anywhere carries a caption naming a textbook as the source of a *figure*. The Epp lineage is real for wording and content — the logical-equivalence names, the valid-argument-form names and the twelve numbered set identities are Epp's — but it is **not** documented evidence for any drawing style. Do not cite Epp as a figure source.

Slide frame and page furniture

Element	Convention	Evidence	Holds?
Header band	Page 1 of each deck: `MATH1061/MATH7861 Discrete Mathematics Semester 2, 2026` at far left, `Lecture N - Title` at far right. Later pages: right-hand lecture label only. A full-width thin black rule sits beneath.	`L02 p1` vs `L02 p2`; identical in every deck L02–L18	Every deck except L01, which has UQ-branded title slides
Pre-work box	A **purple rounded-rectangle** box, top right of page 1, headed `Pre-work video:` / `Pre-work videos:` in bold, listing `Video NNN Title` with three-digit zero-padded numbers	`L02 p1`, `L03 p1`, `L05 p1`, `L07 p1`, `L10 p1`, `L13 p1`, `L16 p1`, `L18 p1`	All 17 decks L02–L18 — the single most consistent element in the course
Learning goals	Bold `Learning Goals` heading, numbered list `1. 2. 3.`	`L02 p1` … `L18 p1`	Every deck L02–L18 (L07 p1 uses an unnumbered list — the one exception)
Student Q&A	Bold blue heading `Student questions and comments:` (or `Student questions:`), questions in **blue**, answers in **red**	`L02 p1`, `L04 p1`, `L06 p1`, `L08 p1`, `L11 p1`, `L12 p1`, `L15 p1`, `L18 p1`	Every deck L02–L18

Element	Convention	Evidence	Holds?
Activities	Bold `Activity N:` restarting at 1 each lecture; pre-class items as `Q1 from the pre-class questions`; Clinch decks add `Poll question N:`	`L02 p3`, `L07 p5`, `L12 p2`, `L17 p3`	Throughout

Figures are **never numbered and never labelled "Figure"**. When a caption exists it sits centred directly beneath the drawing, either as the mathematical expression it depicts (`(A - B)`, `L18 p2`) or as a short prose verdict (`not a function` / `a function`, `L18 p9`).

Colour

Printed colour on slides

- **Red = the thing you must not get wrong.** Warnings, answers to student questions, "IMPORTANT NOTE", "Take care with notation!", "WARNING". `L01 p4` ("hurdles"), `L02 p1`, `L08 p2` ("WARNING"), `L10 p1` ("IMPORTANT NOTE"), `L11 p4` ("Take care with notation!"), `L17 p2` ("Warning!!!!"), `L17 p9`. Holds in both lecturers' decks, all semester.
- **Blue = student voice and explanatory gloss.** Student questions (`L02 p1` onward, universal) and, in Clinch's decks, a plain-English restatement placed *directly beneath* a formal condition written in red — `L12 p3` (`d|a and d|b` in red, "*d is a divisor of both a and b*" in blue below it), `L12 p4` (same pattern for lcm). That paired red-above/blue-below idiom is specific and recurs.
- **Colour as correspondence, not meaning.** Where two expressions must be matched piece-by-piece, matching pieces are given matching colours. `L03 p9` (a three-row table where `a  b`, `a if and only if b`, `b is necessary and sufficient for a` are colour-aligned column by column), `L06 p6` (quantifier negation, below), `L11 p3` (quotient in red above q, remainder in red above r, and the same two colours on the worked examples). This is the one place colour genuinely carries information.
- **Pastel highlight capsules** (yellow, green, blue, pink, orange) mark the single expression being tracked through a calculation — it keeps its capsule colour as it moves line to line. Heavily used by Clinch: `L11 p3`, `L12 p6`, `L13 p6`, `L14 p3`, `L16 p5`, `L17 p5`. Rare in Davies's decks.

Annotation ink (live handwriting)

Both lecturers annotate over the slides in ink. **Blue is the predominant ink for the main worked solution** — `L01 p13`, `L02 p7`, `L04 p4`, `L07 p4`, `L08 p5`, `L10 p5`, `L13 p7`, `L16 p7`, `L18 p3`. Other colours (green, red, purple, orange, teal) are layered over it for justifications, second solutions, and circling.

Be honest about what ink colour means: mostly, nothing fixed. It separates successive layers of annotation for legibility, and rotates. The two places it *is* systematic:

1. **Correspondence** — e.g. `L02 p6`, where each intermediate truth-table column is written in its own colour and numbered in that colour to show evaluation order, with the final column ringed and labelled "final answer".
2. **Verdict marks** — a green circled `T` and a red circled `F` for true/false answers: `L05 p7`, `L05 p8`, `L06 p4`. Green tick / red cross for correct/incorrect options: `L06 p3`, `L09 p4`, `L16 p2`.

Boxes and callouts

Shape	Meaning	Evidence	Holds?
Purple rounded rectangle	A named, quotable object: definition, Fact, Lemma, Theorem — and the pre-work video box	`L07 p1` (the six definitions), `L07 p2` (two Facts), `L08 p2`, `L09 p6` (Fact + Lemma), `L10 p6`, `L10 p7` (Unique Factorisation Theorem), `L10 p9`, `L10 p12`, `L11 p3` (Quotient-Remainder Theorem), `L11 p7`, `L11 p8`	Strong: both lecturers, L07 through L12
Plain thin black rectangle	A structural panel — a proof skeleton, or a two-way comparison	`L07 p3`, `L08 p3`, `L09 p2` (the four proof-method skeletons), `L11 p3` ("2 equivalent ways of defining…")	Consistent where it appears

Shape	Meaning	Evidence	Holds?
Red rectangle	Davies: a fact that must be memorised or a warning. Clinch: a numbered step caption or a named principle	`L02 p8` (De Morgan's laws), `L03 p4`, `L03 p9`, `L04 p1` ("Recall:"), `L01 p4`; and `L10 p6` (margin commentary), `L12 p10` (steps 1/2/3), `L14 p6` (Well-ordering Principle)	Both uses recur; see Inconsistencies
Margin callout with arrow	A red-outlined box in the right margin, with a red arrow to the exact proof line it explains, and a purple at the anchor point	`L10 p6` (two callouts, two arrows, two stars)	Seen once — extrapolating from a single instance
Hand-drawn cloud	An aside or a menu of options: "What type of proof will work?" with / against each method	`L09 p4`, `L09 p5`, `L09 p6`, `L09 p7`; `L14 p3`	Four consecutive slides in L09, once in L14

Logic notation

Fixed by the formula sheet and used unchanged in every deck:

Symbol	Reading	Note
`p`	not p	**The course symbol is the tilde , not ¬.** `L02 p1` answers this directly: "some common symbols for not include and ¬" — both exist, this course writes . Used in every deck and on the formula sheet
`p q`, `p q`	and, or	is **inclusive** or; exclusive or is (`L02 p5`)
`p → q`	if p then q	`L03 p1`; defined as `p q`

Symbol	Reading	Note
p q	if and only if	A **connective that combines** statements
p q, p q	logically equivalent / not	A **relation that compares** statements
t, **c**	tautology, contradiction	Bold upright, lower case. Formula sheet; L02 p9
	therefore	Prefixes the conclusion of an argument

versus is stated explicitly and must not be blurred (L03 p7): "The biconditional ( ) is a conjunction used to **combine** statements (similar to +, -, x, ÷ … with numbers). Logical equivalence ( ) is used when **comparing** two statements (similar to = with numbers)."

Precedence (L02 p5, extended at L03 p7): first; and equal second; → and equal third; within a tier, work left to right — "but parentheses should always be used to avoid confusion."

Statement variables are lower-case italic p, q, r. When a problem is in English, the letters are chosen as mnemonics of the English phrase: w = wages, b = buying, d = depression (L04 p4); q = queued, s = sent to the buffer (L05 p2).

Truth tables

The course calls this **"standard truth table form"** and requires it (L02 p4): *"You should construct your truth tables in standard form so that you are sure you have covered all combinations … and it is easy for someone to check your work."*

- **Values are the single upright capitals T and F** — never 1/0, never True/False. Universal: L02 p4, L03 p1, L04 p5, L06 p2, L07 p3, L09 p2.
- **2 rows for *n* variables**, in a fixed order: the first variable is T for the whole top half then F for the whole bottom half; each subsequent variable halves again; the last variable alternates T,F every row. Shown for n = 1, 2, 3, 4 side by side at L02 p4.
- **A narrow solid-filled amber/orange column separates the input block from the compound-form block.** L02 p4 (after the variable columns, before "Statement form"); L04 p5 (after the variables *and* their simple negations, before the premise and conclusion columns). Two instances, both lecturers' style aside — treat as the course's own table furniture.
- **Header row typeset in red italic maths** (p, q, p → q), body values in black upright. L03 p1, L03 p4, L04 p1, L04 p2, L04 p5, L07 p3, L08 p3, L09 p2. Consistent throughout Davies's decks; Clinch draws no printed truth tables.
- Full ruled grid — every cell bordered.

- **When built by hand, columns are added left to right in evaluation order and numbered beneath**, each in its own ink colour, with the answer column ringed. L02 p6, L03 p2. At L03 p2 the two columns that must match are marked with a beneath and the verdict written out: "Starred columns are identical so ((p q) (p q)) p".

Arguments and inference proofs

Argument form (formula sheet; L04 p2, L04 p3, L04 p6, L05 p2, L05 p4): premises stacked one per line, left-aligned; the conclusion on the last line prefixed by . **No horizontal rule above the conclusion.** Premises are numbered 1. 2. 3. and the line is left unnumbered.

Rules-of-inference proof (L04 p6, L04 p7, L05 p4): numbered lines down the left, the derived statement in the middle, and the **justification in its own right-hand column**, phrased as from by contraposition, from & by transitivity, by double negative law, by definition of →. The annotation at L04 p6 is emphatic: **"must write explanations!"**.

Equivalence-law chains (L02 p10, L03 p8, L18 p5, L18 p7): first line is the starting expression; every subsequent line begins with (or = for sets), vertically aligned, with the named law in a right-hand column — by de Morgan's law, by the negation law, by identity law. For set identities the numbered law from the formula sheet is cited: by set diff law (12), by de Morgan (9b), dist. law (3b), complement law (5b), identity law (4a) (L18 p7).

Biconditional proofs are marked by direction: and label the two halves (L12 p8). Two named methods, with a stated preference — Method 1, prove each direction separately, is annotated "**recommended method**"; Method 2, an iff chain, is annotated "**Be careful! Quicker method but easy to make a logical error**" (L12 p8, L12 p9).

Quantifiers

- x D, P(x) — universal, **comma** after the domain.
- x D such that P(x) — existential, the words **"such that"**, abbreviated **s.t.** (L06 p1, L05 p7).
- Both established at L05 p5 and used unchanged through L08 p1, L17 p3.
- reads "is an element of" / "belongs to"; "is not an element of" (L05 p5).
- Negation rules, stated as displayed equivalences at L08 p1: (x D such that P(x)) x D, P(x) and (x D, P(x)) x D such that P(x).

Number sets

Blackboard-bold, established at `L05 p1` and used unchanged to the end: , , , .

Irrational numbers are written ¯ — with an overbar (`L05 p1`, `L09 p1`, `L09 p3`, `L09 p6`). Not ; the equality = ¯ is written by hand at `L09 p1` as a gloss, but the barred form is what the printed slides use.

Restrictions are written as superscripts on the set symbol, in italic, not as subscripts and not as prose:

```
^even ^odd ^nonneg ^prime ^composite ^{>1} ^{ 2} ^{ 1}
```

Evidence: `L05 p1` (, ^even, ^nonneg), `L07 p2` (^even, ^odd, ^prime, ^composite, ^{ 1}, ^{>1}), `L07 p6` (^{ 2}), `L10 p12` (^{ 2}), `L17 p10` (,). Strong and semester-long.

Proof layout

The four methods each have a **fixed printed skeleton**, given verbatim on a repeated slide (`L07 p3`, `L08 p3`, `L09 p2` — the same panel, grown one box at a time as methods are introduced). Reproduce these wordings:

- **Direct proof** — `Let x D and suppose P(x). … Hence Q(x).`
- **Disproof by counterexample** — `Let x = . Then x D and P(x) is true. However, Q(x) is false. So x is a counterexample and the statement is false.`
- **Proof by contradiction** — `Suppose the negation of x D, P(x) → Q(x) is true. That is, suppose x D such that P(x) is true but Q(x) is false. … Hence something that is obviously wrong… This is a contradiction, and we conclude that … is true.`
- **Proof by contraposition** — `We prove the contrapositive of x D, P(x) → Q(x), namely that x D, Q(x) → P(x). Let x D and suppose Q(x). … Hence, P(x).`

Each box carries a red caption directly beneath it saying what it establishes: `Shows the statement is true.` / `Shows the statement is false.` (`L07 p3`, `L08 p3`).

Other layout facts:

- `Proof:` **label**, then the hypothesis line, then the working with = signs vertically aligned in a column, then a `Hence`/`Thus`/`So` conclusion. `L07 p4`, `L07 p5`, `L08 p5`, `L09 p3`, `L10 p5`.
- **Method comparison is a two-column layout** with bold centred headings — `Direct Proof` left, `Proof by contradiction` right (`L08 p5`, `L08 p6`); `Proof SMI` / `Proof MI` / `Proof WOP` (`L14 p7`, `L14 p8`).

- **Without loss of generality, spelled out then abbreviated WLOG** (L08 p8, L09 p5).
- **Case splits are labelled Case 1: / Case 2:** (L10 p6, L10 p12, L14 p5).
- **Every proof unfolds definitions.** The six definitions in the purple box at L07 p1 / L08 p2 are stated as **biconditionals using** — e.g. n is an even integer n = 2k for some integer k, d|n d 0 and n = dk for some integer k — because they are used in both directions.
- **Introduce a *new* variable for each new existential.** Annotated explicitly at L08 p6: "use a **new** variable!" where m = 2k+1 is followed by n = 2 +1.

End-of-proof mark

Printed proofs end with an **open square** : L07 p3, L07 p4, L08 p3, L09 p2, L10 p6, L10 p11, L10 p12.

Handwritten proofs end with either a hand-drawn open square (L07 p5, L08 p8, L09 p4, L09 p5, L09 p6) or a **double slash //** (L10 p5, L11 p6, L13 p7, L13 p9, L14 p10, L15 p5, L16 p7, L18 p3, L18 p7). At L11 p6 the lecturer writes an explicit legend of the acceptable marks: **// or QED**. Treat all four as sanctioned.

Contradiction mark

Davies writes ( ) — two arrows meeting — inline at the point the contradiction lands: L08 p3, L08 p5, L08 p6, L08 p7, L08 p8, L09 p6. Clinch instead uses a small crossed glyph inline, e.g. "(proof by)" for "proof by contradiction" (L14 p6, twice). See Inconsistencies.

Induction

The named-step skeleton is fixed and printed identically in five separate proofs:

```
Proof:  Let P(n) be the predicate  statement .
Basis step:        verify P(a)  … so P(a) is true.
Inductive step:   Prove that  k    with k  a, if P(k) is true, then P(k+1) is true.
Inductive hypothesis:  Suppose that k  a is an integer and that P(k) is true.
        LHS of P(k+1):  chain of = with the RHS reached
        Therefore P(k+1) is true.
Therefore, by  PMI / SMI ,  statement  for each integer n  a.
```

Evidence: L13 p6, L13 p8, L14 p3, L14 p7, L16 p6, L16 p8, L16 p10. Step labels are printed in **blue** at L13 p6 and in **red** at L14 p3/L14 p7.

- `LHS` / `RHS` are the standard abbreviations, and the two sides are named and worked separately: `LHS of P(1): … RHS of P(1): …` (`L13 p6`, `L13 p7`, `L16 p7`).
- The step where the hypothesis is used is annotated `(by the inductive hypothesis)` on the same line (`L13 p6`, `L13 p7`, `L14 p3`).
- Slides ask for the steps to be named: *"It is a good idea to list these explicitly in your proofs to help the reader to follow the proof"* (`L13 p1`).
- Multiple base cases are listed one per line when strong induction needs them (`L14 p4`, `L15 p8`).

Number-theory notation

- **Divides: `d | n`. Does not divide: `d  n`** (slashed bar). `L05 p7`, `L07 p1`, `L10 p1`, `L10 p10`, `L14 p8`.
- Equivalent phrasings given at `L10 p1`: *d divides n*, *d is a divisor of n*, *d is a factor of n*, *n is a multiple of d*.
- **`n mod d = r`** (upright `mod` as a binary operator, no brackets) versus **`m  n (mod d)`** (congruence, `(mod d)` bracketed after the statement). These are different things and the course flags it twice under a red heading `Take care with notation!` — `L11 p3`, `L11 p4`, recapped `L12 p2`.
- **`gcd(a, b)` and `lcm(a, b)`**, lower case, upright roman (`L12 p3`, `L12 p4`). At `L12 p3` the alternative `hcf(a,b)` is written and then **struck through** — it is not this course's name.
- **Floor `x` and ceiling `x`** (`L11 p2`).
- **Standard factored form**: `n = p^{e} p^{e}  p_k^{e_k}` with the primes in **increasing** order `p < p <  < p_k` (`L10 p7`, annotated "increasing" at `L10 p8`).
- **Multiplication is a centred dot** · in printed slides: `2 · 3 · 5 · 7 · 11  p` (`L10 p11`), `5859 = 1232 · 4 + 931` (`L12 p6`), `2 · 100 + 5 · 10 + 4 · 1` (`L11 p8`), `3 · 1 + 2` (`L13 p6`).

Sequences, sums and recurrences

- **Summation `Σ` with `i = m` below and `n` above**; index letters `i`, `j`, `k`; change of variable written as `j = i + 1` (`L13 p3`, `L13 p4`, `L13 p6`).
- A recursive definition of a **sequence** requires two named parts: **a recurrence relation** and **initial conditions** (`L15 p3`, `L16 p7`, `L16 p9`).
- A recursive definition of a **set** requires three named parts: **Base**, **Recursion**, **Restriction** — always in that order, with the Restriction worded "Nothing is in B other than objects defined by the Base and the Recursion" (`L15 p6`, `L15 p7`).
- Verifying a guessed explicit formula has two named methods: **Method 1** (check the initial conditions and the recurrence relation — annotated "quicker") and **Method 2** (prove it by induction — annotated "rigorous"). `L16 p4`, used at `L16 p5`–`L16 p10`.

- `n!` with `0! = 1` by definition (`L15 p5`).

Set-theory notation

Notation	Meaning	Evidence
`x  A`	is an element of	`L17 p1`
`A  B`	is a subset of	`L17 p1`, `L17 p3`
`A  B`	is a **proper** subset of (and there is an element of B not in A)	`L17 p3` — the course does use strictly for proper
`A  B`, `A  B`	intersection, union; indexed forms `_{i=1}^{3} A` , `_{i=1}^{3} A`	`L17 p1`, `L17 p6`
`A - B`	set difference; **`A \ B` is stated to be equally acceptable**	`L17 p1` — answered directly: "Yes. `A - B` can be written as `A \ B`"
`A`	complement, superscript `c`	`L17 p1`, `L18 p1`, formula sheet
`(A)`	power set, calligraphic script `P`	`L17 p1`, `L17 p7`, `L17 p8`
`A × B`	Cartesian product; `×` also written ²	`L17 p1`, `L17 p9`
	empty set, also written `{ }`	`L17 p5`
`|A|`	cardinality	`L17 p7`, `L17 p9`
`{n  | n  2 (mod 8)}`	set-builder with a vertical bar	`L17 p3`, `L18 p2`
`[a, b]`	closed interval `{x  | a  x  b}`	`L17 p6`

Sets take capital italic letters (`A, B, C, S, T, D, E, R, X, Y`), the universal set is `U`, elements take lower case (`x, y, t`). `L17 p3`, `L18 p1`, `L18 p6`.

The / distinction is the course's flagged trap — a red `Warning!!!!` at `L17 p2` and the first learning goal of both L17 and L18. `2  {2}` (`L17 p2`); is false, is true (`L17 p1`, `L17 p2`).

Proof of set statements has three named methods (`L18 p4`): **the element method**, **using the laws of logical equivalence**, and **using set identities** (citing the formula sheet's numbering, as above).

Functions (`L18 p9`–`L18 p11`): `f: X → Y`; `f  X × Y` such that for each `x  X` there is a unique `y  Y` with `(x, y)  f`. Domain `X`, **co-domain** `Y` (hyphenated in print), range `f(X) = {f(x)`

`: x  X}`. Image of a subset `f(A)`; **inverse image or preimage** `f ¹(B)` — both words are given as correct.

Diagrams

This is what the decks actually draw. Nine idioms recur; everything else is a one-off and is labelled as such.

1. Nested closed curves for containment (Euler style)

Number systems are drawn as **concentric plain closed curves with each label written inside its own region** — `: rational numbers` outermost, containing `: integers`, containing `: natural numbers`; labelled above the frame, `¯: irrational numbers` in a separate adjacent panel. Thin blue rule dividing the two panels; curves thin black; no shading; no element dots. `L05 p1`. **Seen once for number systems**, but the same nested-oval idiom recurs for subsets at `L17 p3` and `L18 p3`, so the shape convention is safe — the specific number-system figure is not.

2. Rectangle partitions

A **plain rectangle with a thin blue/teal outline** stands for the ambient set; **thin internal straight lines** split it into disjoint parts; each part is labelled in **red italic** with its superscript-qualified name; the ambient set's own label sits **above and outside** the top-left corner. Two instances on one page: split into `^even | ^odd`, and split into `^{ 1} | ^prime / ^composite`. `L07 p2`. **Seen on one page only** — but it is the course's only *printed* partition figure, and it is the natural pairing with the superscript set notation above.

3. Two-circle Venn diagrams

- **Exactly two circles of equal radius, overlapping, drawn side by side in a horizontal row**; thin black outline; labels `A` and `B` placed **inside** each circle near its top outer edge.
- **The region being denoted is filled solid grey.** The undecorated pair is drawn first, unshaded, as the reference.
- **The expression it depicts is the caption, centred directly beneath** — `(A - B)`, `(B - A)`.
- `L18 p2`. Three diagrams in a row.
- Hand-drawn variant at `L18 p3`: a **rectangle for the universe with red diagonal hatching** over the complement region, with `A` (green) inside `B` (green outline) inside it.

Critical rule, stated by the course itself: at `L18 p2` a hand annotation brackets all three Venn diagrams and writes "**not a proof!**". A Venn diagram in this course illustrates a set statement; it never discharges one. Proofs go through the element method, the laws of logical equivalence, or the set identities (`L18 p4`).

4. Small icon strip for set relations

A row of thumbnail two-set diagrams, one per notation, aligned under the list of symbols they gloss: **set A drawn as a green ellipse, set B as a pale blue/grey ellipse**; an element as a small filled dot; the operation's region filled or hatched in blue. `L17 p1` (the full strip: `x  A`, `A  B`, `A  B`, `A  B`, `A  B`, `A - B`, and `A = B` drawn as coincident ellipses). Repeated as single icons at `L17 p3`, `L17 p4`, `L17 p6`. **The green-A / grey-B colour pairing holds across all five appearances.**

5. Function arrow diagrams (bipartite)

- **Two vertical ellipses side by side: the domain `X` on the left, the co-domain `Y` on the right**, each labelled **beneath** the ellipse in blue.
- Elements are **labelled dots stacked vertically inside** each ellipse.
- Each element of `X` is joined to its image by a **straight blue line segment**. Flow is left to right.
- A subset under discussion is marked by a **translucent coloured blob** drawn around the relevant dots, labelled just outside in the matching colour with its expression (`f({2,3})`, `f ⁻¹({1})`, `f({1})`).
- `L18 p11` — four diagrams on the page, all hand-drawn, all following this arrangement.

6. Cartesian-plane graphs

Axes as **thin double-headed arrows crossing at the origin**; the `y` label at the top of the vertical axis and the `x` label at the right of the horizontal axis, both immediately beside the arrowhead; the curve in **blue**; marked points as **filled dots labelled with their coordinate pair** `(0,1)`, `(5,11)`; a **prose caption centred beneath** (`Plot of the function y = 2x + 1.`, `not a function`, `a function`). The vertical line test is shown as a **red vertical line** crossing the curve twice. `L18 p9` — three panels in a row.

7. Number lines

Drawn as a **horizontal line with an arrowhead at each end**, evenly spaced tick marks, **integer labels beneath the ticks**, and marked points or brackets **above** the line. `L11 p2` (two of them: the real line from −5 to 4 for floor/ceiling, and a minimal one showing only `k` and `k+1` with `k + ½` between); `L17 p6` (interval `[a,b]` drawn above the line with filled dots at the closed endpoints, the axis labelled at the right end).

8. Colour-coded correspondence diagrams

The single most information-dense idiom in the course. At `L06 p6`, the negation of a two-quantifier statement is shown as **two copies of the same statement schema stacked one above the other and horizontally aligned part-for-part**, with each corresponding pair of parts underlined in its own colour and joined by a **double-headed arrow of that colour**:

- red: `x  X  x  X such that`
- green: `y  Y such that  y  Y,`
- cyan: `P(x,y)  P(x,y)`
- a purple curved double-headed arrow spanning both rows, **labelled `negation`**

Seen once, but it is the canonical figure for the topic. The same alignment-plus-colour technique without arrows appears at `L03 p9` and `L12 p3`.

9. Algorithm and process cascades

- **Euclidean algorithm** (`L12 p6`): the successive equations `a = bq + r` stacked vertically down the left; each numeral wrapped in a **pastel capsule, and each number keeps the same capsule colour as it migrates** from `b` to `a` and `r` to `b` on the next line; **red diagonal arrows** trace those two migrations; a parallel right-hand column carries the matching `gcd(a,b) = gcd(b,r)` chain with the same capsule colours. Colour here is pure identity-tracking.
- **Sequence hops** (`L16 p2`, `L16 p3`, `L15 p5`): terms written in a horizontal row, **short curved arrows above or below each gap, labelled with the operation** (`+3`, `×3`, `× −1`).
- **Recursive-construction tree** (`L15 p6`): root at the **left**, branches as plain straight segments fanning **rightwards**, node labels at the branch tips, **no arrowheads on the branches**. The separate derivation chain uses arrows with the rule name written above: `--1--> 1 --1--> 11 --01--> 1101` (`L15 p7`).
- **State sequences** (`L16 p11`, Tower of Hanoi): three vertical blue pegs on a base line; discs as short red horizontal bars, widest at the bottom; **each state enclosed in a plain black rectangle**; states run **left to right**; green curved arrows between states labelled with the move count (`m_{n-1}`, `1`, `m_{n-1}`).
- **Partitions of a set as coloured blobs** (`L17 p10`): irregular non-overlapping closed curves in distinct colours, each containing dots, each **labelled just outside in its own colour** (`A` green, `A` red, `A` yellow, `A` blue). On the same page a number line is partitioned instead by **weaving highlighter stripes** — pink for `S`, green for `T` — over the tick labels.

Tables

- Full ruled grid, header row present.

- Truth tables as described above.
- Comparison tables put the item varying across the top and the cases down the side, with `T/F` in the cells (`L06 p2`: three quantified statements across, three sets `S` down).
- **Material reproduced from the examination formula sheet is pasted as an image and therefore appears in serif LaTeX type**, visually distinct from the sans-serif slide body: `L02 p9` (Logical Equivalences), `L04 p3` (Valid Argument Forms), `L18 p6` (Set Identities). If reproducing one of those tables, reproduce the formula sheet's own wording and numbering.

Inconsistencies

The course does these things more than one way. **Both readings are recorded; no winner is picked. A decision is needed before any new figure fixes one.**

1. **End-of-proof mark.** Printed proofs use (`L07 p3`, `L10 p11`). Handwritten proofs use a hand-drawn square (`L09 p4`) *or* `//` (`L13 p7`, `L18 p7`). At `L11 p6` the lecturer offers all of `//`, , and `QED` as equivalent. Clinch's printed induction proofs (`L14 p7`, `L16 p6`) end with a concluding sentence and **no mark at all**. Four options, all sanctioned. **Decide.**

2. **Contradiction mark.** Davies writes `( )` (`L08 p3`, `L08 p5`, `L08 p6`, `L08 p7`, `L08 p8`, `L09 p6`). Clinch writes a crossed glyph and uses it as a noun — "(proof by)" (`L14 p6`, two occurrences on one page). **Decide.**

3. **Name of the induction principle.** Within four decks: "by the principle of mathematical induction" (`L13 p6`), "by PMI" (`L13 p8`, `L16 p12`), "by MI" (`L16 p10`), "by SMI" (`L14 p5`, `L16 p8`), "by PSMI" (`L14 p3`, `L14 p7`). PMI/MI and SMI/PSMI are used for the same two principles in the same lecturer's decks. **Decide.**

4. **Sequence delimiters.** Printed: `{b }_{n 0}` (`L13 p2`), `{a }_{n 1}` (`L15 p3`), and bare `a , a , a , …` (`L14 p3`, `L16 p2`). Handwritten on the same slide (`L13 p1`): `(a )_{n 0}` **and** `{a }_{n=1}^{ω}`. Four forms. **Decide.**

5. **Set-builder separator.** Vertical bar in `{n  | n  2 (mod 8)}` (`L17 p3`), `{(x,y)|x  and y  }` (`L18 p9`); colon in `f = {(x,y)  ×  : y = 2x + 1}` (`L18 p9`, same page), `f ¹(B) = {x  X : f(x)  B}` and `f(A) = {f(x): x  A}` (`L18 p11`). Both on the same page as each other. **Decide.**

6. **Multiplication sign.** Printed slides use the centred dot: `1232 · 4` (`L12 p6`), `2 · 3 · 5 · 7` (`L10 p11`). Handwriting on the very next slide uses `×`: `2431 × 1` (`L12 p7`), `4 × 9` (`L13 p9`), `×3` (`L16 p2`). **Decide.**

7. **Set difference symbol.** `A - B` throughout the printed slides and the formula sheet; `A \ B` explicitly declared equally correct at `L17 p1`. Not a conflict of fact, but a live choice.

8. **Which box shape carries a named principle.** Purple rounded rectangle for Theorem/Lemma/Definition/Fact (L07 p1, L10 p7, L11 p3) versus a **red** rectangle for the Well-ordering Principle (L14 p6). Same category of object, two shapes.
9. **Text colour inside the purple definition box.** Red at L07 p1, L08 p2, L11 p7; blue at L10 p7 (Unique Factorisation Theorem); black at L07 p2 (the two Facts). Three colours for the same box type.
10. **Capitalisation of the induction step labels.** "Inductive hypothesis" (L13 p6, L14 p3) versus "Inductive Hypothesis" (L16 p6, L16 p8, L16 p10). Also "Let P(n) **be** the predicate" (L13 p7, L16 p6) versus "Let P(n) **denote** the predicate" (L13 p6).
11. **LHS versus L.H.S.** L.H.S / R.H.S with stops at L13 p1; LHS / RHS without at L13 p6, L13 p7, L16 p7, L16 p9.
12. **Does contain 0?** The printed diagram at L05 p1 shows : natural numbers nested inside without saying which. The handwritten annotation on the same slide gives both = {0,1,2,…} **and** = {1,2,3,…} with the note "**take care!**" — the course is deliberately flagging the ambiguity rather than settling it. ^nonneg = {0,1,2,3,…} is defined unambiguously on the same slide and is the safe alternative.
13. **Writing two variables in the same domain.** Three forms shown side by side at L06 p5: x D and y E … (annotated "more clear"), x D and y E …, and x, y D … (annotated "short hand for same domain"). Preference is hinted at but not stated as a rule.

Not established by the source

Be candid: these are where a new diagram would be inventing, not conforming.

- **Arrowheads on function-diagram connectors.** Every mapping at L18 p11 is drawn with a **plain line segment**. Whether an arrowhead is intended, required, or merely omitted in fast handwriting cannot be told from one page. Nothing in the corpus shows an arrowheaded mapping diagram.
- **Three-set Venn diagrams.** Only two-circle Venns are ever printed (L18 p2). The set identities involving three sets (associative, distributive — L18 p6) are never illustrated. There is no evidence for the standard three-circle arrangement, region labelling, or which regions get shaded.
- **A universe rectangle in printed Venn diagrams.** L18 p2 has no enclosing rectangle at all. The only universe rectangle appears in a hand annotation (L18 p3) with red diagonal hatching. Whether a printed figure should include U is unsettled.
- **Shading colour and fill style.** One printed instance uses solid mid-grey (L18 p2); the icon strip uses blue hatching and blue solid interchangeably on the same row (L17 p1). No stated palette.

- **Line weights, stroke colours for plain geometry, and typeface.** Nothing is specified anywhere. Slide body text is a Calibri-like sans; reproduced formula-sheet extracts are serif LaTeX. Beyond that, invent nothing.
- **Figure numbering and cross-referencing.** No figure in 176 pages carries a number or the word "Figure". There is no established way to refer to a diagram from body text.
- **Any convention for the second half of the course.** L19–L36 have no decks in this set. That means **nothing is established** for: one-to-one / onto / inverse function diagrams; composition diagrams; cardinality and countability arguments (the Schröder–Bernstein theorem is on the formula sheet but never drawn); **relation digraphs**; **Hasse diagrams** for partial orders; equivalence-class pictures; Cayley tables for groups; counting trees and Pascal's triangle; **graph drawings** (vertex shape, edge style, labelling, directedness); walks/trails/circuits notation; **adjacency-matrix layout**; and tree drawings for graph theory. The only graph-theoretic images in the corpus are the Königsberg bridge photograph and a Tower of Hanoi photograph on `L01 p13`, which establish nothing.
- **Whether hand-annotation colour should be reproduced at all.** The ink layers exist because the slides are annotated live. A clean redrawn figure has no equivalent of "second colour for the second pass", and the corpus gives no printed-only version of most annotated diagrams to copy from.
- **A logic-gate or circuit notation.** None appears; MATH1061 is not a circuits course, but `L01 p10` motivates logic via computer science, so the absence is worth stating.
- **Whether may ever mean plain subset.** The course defines it as *proper* subset (`L17 p3`) and uses it that way consistently in the pages seen, but never says "we never use loosely", so a reader coming from other texts has no explicit prohibition to point at.

Practice Problems — the Semester-wide Set

This is the course's one problem set for the whole semester — a single document rather than a per-week handout. 46 sections, one per pre-work video, each with the relevant textbook pages, a set of practice problems and full worked solutions. Everything is reproduced here in full, so you never need to open the PDF.

Applied Classes start in **Week 2** and run to Week 13 — twelve of them, matching the "best 8 of 12" Applied Classes component worth 30% (see math1061[69]). Each week's class applies the *previous* week's lecture content, so the material a class covers runs roughly one week behind the lecture column below. Which sections a given Applied Class draws on is not stated in any of the source material.

These problems belong to the roughly three hours of individual study each week, which are for revising the videos, readings and classes **and completing the Practice Problems**. They are the out-of-class half of the course, and they are *not* the same thing as the Assessed Exercises,

[69] courses/math1061/math1061.pdf

which are done and submitted inside the class itself. (That class goes by two names, the **Active Class** and the **Applied Class**. They are one session.)

How each section is laid out

1. Relevant textbook pages for Epp 4th edition and 5th (metric) edition
2. **Practice problems**
3. **Solutions** — full worked solutions, not just answers
4. Occasionally a non-assessable **Application** block (see the table below)

Errata

The source document carries an errata page listing corrections made during semester. **Last update: 24 August 2026**, with two entries:

- On page 195, the solution to 1(b) was corrected (the note in brackets had previously copied the note from 1(a)) — already applied in §45 below.
- **Question 2 in section 30 has been updated, and the original version moved to section 32** (because it refers to antisymmetry) — applied in §30 and §32 below.

If something looks wrong, post on the Ed Discussion board.

Section index

Every section is below, with its topic and textbook page ranges. Each section goes with one pre-work video (§1 with *Video 001*, and so on up to *Video 046*), and the Lecture (week) column places it in the schedule of math1061[70] by that video number. The Notes column links to the concept note covering that material where one exists; the rest are still ahead in the semester.

§	Topic	Lecture (week)	4th ed.	5th ed.	Notes
1	Logical Form	L2 (wk 1)	023–029	037–043	logical-connectives[71]
2	Logical Equivalence	L2 (wk 1)	030–038	043–053	logical-equivalence-laws[72]

[70] courses/math1061/math1061.pdf
[71] courses/math1061/logical-connectives.pdf
[72] courses/math1061/logical-equivalence-laws.pdf

§	Topic	Lecture (week)	4th ed.	5th ed.	Notes
3	Conditional Statements	L3 (wk 1)	039–050	053–066	conditional-statements[73]
4	Valid and Invalid Arguments	L4 (wk 2)	051–063	066–079	valid-argument-forms[74]
5	Methods for Determining Validity	L5 (wk 2)	051–063	066–079	determining-argument-validity[75]
6	Quantified Statements	L5 (wk 2)	096–108	108–121	quantified-statements[76]
7	Negation of Quantified Statements	L6 (wk 2)	108–117	122–131	quantified-statements[77]
8	Statements with Multiple Quantifiers	L6 (wk 2)	117–131	131–146	quantified-statements[78]
9	Direct Proofs and Counterexamples	L7 (wk 3)	145–163	160–182	proof-techniques[79]
10	Proof by Contradiction	L8 (wk 3)	198–207	218–228	proof-techniques[80]
11	Proof by Contraposition	L9 (wk 3)	198–207	218–228	proof-techniques[81]
12	Rational Numbers	L9 (wk 3)	163–170	183–190	rational-and-irrational-numbers[82]
13	Divisibility	L10 (wk 4)	170–179	190–199	divisibility-and-factorisation[83]

[73] courses/math1061/conditional-statements.pdf
[74] courses/math1061/valid-argument-forms.pdf
[75] courses/math1061/determining-argument-validity.pdf
[76] courses/math1061/quantified-statements.pdf
[77] courses/math1061/quantified-statements.pdf
[78] courses/math1061/quantified-statements.pdf
[79] courses/math1061/proof-techniques.pdf
[80] courses/math1061/proof-techniques.pdf
[81] courses/math1061/proof-techniques.pdf
[82] courses/math1061/rational-and-irrational-numbers.pdf
[83] courses/math1061/divisibility-and-factorisation.pdf

§	Topic	Lecture (week)	4th ed.	5th ed.	Notes
14	Modular Arithmetic	L11 (wk 4)	482–485	528–531	modular-arithmetic[84]
15	The Euclidean Algorithm	L12 (wk 4)	220–224	250–254	gcd-lcm-and-euclidean-algorithm[85]
16	Sequences	L13 (wk 5)	227–244	258–275	sequences-and-summations[86]
17	Mathematical Induction	L13 (wk 5)	244–268	275–301	mathematical-induction[87]
18	Strong Induction and the Well Ordering Principle	L14 (wk 5)	268–279	301–314	strong-induction-and-the-well-ordering-principle[88]
19	Recursive Definitions	L15 (wk 6)	290–304	325–340	recursive-definitions[89]
20	Solving Recurrence Relations	L16 (wk 6)	304–328	340–364	solving-recurrence-relations[90]
21	Set Theory Definitions	L17 (wk 6)	336–351	377–391	sets-and-set-notation[91]
22	More Definitions and Examples of Sets	L17 (wk 6)	336–351	377–391	sets-and-set-notation[92]
23	Properties of Sets	L18 (wk 7)	352–366	391–407	proving-set-properties[93]

[84] courses/math1061/modular-arithmetic.pdf
[85] courses/math1061/gcd-lcm-and-euclidean-algorithm.pdf
[86] courses/math1061/sequences-and-summations.pdf
[87] courses/math1061/mathematical-induction.pdf
[88] courses/math1061/strong-induction-and-the-well-ordering-principle.pdf
[89] courses/math1061/recursive-definitions.pdf
[90] courses/math1061/solving-recurrence-relations.pdf
[91] courses/math1061/sets-and-set-notation.pdf
[92] courses/math1061/sets-and-set-notation.pdf
[93] courses/math1061/proving-set-properties.pdf

§	Topic	Lecture (week)	4th ed.	5th ed.	Notes
24	Functions Defined on General Sets	L18 (wk 7)	383–396	425–439	functions-on-general-sets[94]
25	one-to-one, Onto, and Inverse Functions	L19 (wk 7)	397–416	439–461	
26	Composition of Functions	L20 (wk 7)	416–427	461–472	
27	Cardinalities	L21 (wk 8)	428–441	473–486	
28	Countable and Uncountable Sets	L22 (wk 8)	428–441	473–486	
29	Relations on Sets	L23 (wk 8)	442–449	487–494	
30	Reflexivity, Symmetry, Transitivity	L23 (wk 8)	449–459	495–505	
31	Equivalence Relations	L24 (wk 9)	459–478	505–523	
32	Partial Order Relations	L25 (wk 9)	498–515	546–563	
33	Definitions and Examples of Groups	L26 (wk 9)	—	—	
34	Elementary Properties of Groups	L27 (wk 10)	—	—	
35	Group Isomorphisms	*not scheduled*	—	—	
36	Definitions and Examples of Fields	L28 (wk 10)	—	—	
37	Introduction to Counting	L29 (wk 11)	516–553	564–604	

[94] courses/math1061/functions-on-general-sets.pdf

§	Topic	Lecture (week)	4th ed.	5th ed.	Notes
38	Counting Selections	L30 (wk 11)	516–553	564–604	
39	Introduction to Probability	L31 (wk 12)	516–553	564–604	
40	Binomial Coefficients	L31 (wk 12)	565–591	617–641	
41	Inclusion Exclusion	L32 (wk 12)	545–549	595–599	
42	The Pigeonhole Principle	L33 (wk 12)	554–565	604–616	
43	Introduction to Graph Theory	L33 (wk 12)	625–660	677–697	
44	Walks, Trails and Circuits	L34 (wk 12)	625–660	677–697	
45	Matrix Representations of Graphs	L35 (wk 13)	661–675	698–712	
46	Trees	L36 (wk 13)	683–701	720–742	

§35 Group Isomorphisms has no lecture in the published schedule — the Week 10 row jumps from L27 (V34) to L28 (V36). The section itself carries no practice problems and is marked *optional and will not be assessed*, so the gap is deliberate.

§28 Countable and Uncountable Sets carries its own note in the source: *this content will not be assessed in MATH1061.* It still has a full set of problems and solutions.

Sections 33–36 (groups and fields) list no textbook page ranges.

Non-assessable application blocks

Eight sections carry an extra block of problems, **not assessable for MATH1061**, mostly drawn from K. Rosen, *Discrete Mathematics and its Applications*, 7th ed. (1991), with two from Epp. They connect the pure material to computer science, and are reproduced in full alongside their sections.

After §	Block	Source
2	Applications to Computer Science — bit strings and logic circuits	Rosen pp. 20–24
5	Applications of Valid and Invalid Arguments	Rosen pp. 48–68
9	Applications of Proofs to Computer Science — resolution and automated theorem proving	Rosen pp. 74–80
15	Trace Tables for Algorithms — the Division and Euclidean algorithms	Epp §4.8 (4th ed.) / §4.10 (5th ed.)
21	Computer Representations of Sets	Rosen pp. 134–138
28	Computability of Functions	Rosen pp. 170–177
31	Identifiers — C identifiers as an equivalence-relation example	Rosen pp. 608–618
37	Runtime — counting basic instructions	Epp 5th ed. pp. 787–799

A note on the diagrams: the source PDF draws logic circuits, graphs and Hasse diagrams as figures. Those are reproduced here as explicit gate netlists and edge lists, which carry the same information in a form that renders everywhere.

§1 — Logical Form

Epp 4th ed. pp. 023–029 · 5th ed. (metric) pp. 037–043. Lecture 2 (week 1). Notes: logical-connectives[95].

Problems

1. Write a truth table for the statement form $p \wedge (\sim q \vee r)$.
2. Suppose you know that $(\sim p \wedge q) \vee p$ is false. What can you conclude about the truth values of each of the two variables?

[95] courses/math1061/logical-connectives.pdf

3. Let p be the statement "DATAENDFLAG is off," q the statement "ERROR equals 0," and r the statement "SUM is less than 1,000." Express the following sentences in symbolic notation.

   (a) DATAENDFLAG is off, ERROR equals 0, and SUM is less than 1,000.

   (b) DATAENDFLAG is off but ERROR is not equal to 0.

   (c) DATAENDFLAG is off; however, ERROR is not 0 or SUM is greater than or equal to 1,000.

   (d) DATAENDFLAG is on and ERROR equals 0 but SUM is greater than or equal to 1,000.

   (e) Either DATAENDFLAG is on or it is the case that both ERROR equals 0 and SUM is less than 1,000.

Solutions

1. The truth table is

p	q	r	$\sim q$	$\sim q \vee r$	$p \wedge (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	T	F
F	F	F	T	T	F

2. We know $(\sim p \wedge q) \vee p$ is false. A disjunction is false only when both parts are false, so we must have

$$p \text{ is false} \quad \text{and} \quad (\sim p \wedge q) \text{ is false.}$$

   From p being false, it follows that $\sim p$ is true. Now $(\sim p \wedge q)$ can be false only if q is false (since $\sim p$ is already true). Hence q is false.

   Therefore p **is false and** q **is false**.

3. (a) $p \wedge q \wedge r$.

   (b) $p \wedge \sim q$.

(c) $p \wedge (\sim q \vee \sim r)$.

(d) $\sim p \wedge q \wedge \sim r$.

(e) $\sim p \vee (q \wedge r)$.

§2 — Logical Equivalence

Epp 4th ed. pp. 030–038 · 5th ed. (metric) pp. 043–053. Lecture 2 (week 1). Notes: logical-equivalence-laws[96]*.*

Problems

1. (a) Use the laws of logical equivalence to show that $p \wedge q \equiv \sim (\sim p \vee \sim q)$.

   (b) Use the laws of logical equivalence to show that $\sim (p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p$.

2. For each of the following, write down a truth table for the statement, and determine whether the statement is a tautology, a contradiction, or neither.

   (a) $((p \wedge q) \vee (q \wedge r)) \vee \sim q$

   (b) $(\sim p \vee q) \vee (p \wedge \sim q)$

3. Use De Morgan's laws to write sentences equivalent to the following:

   (a) It is not true that I am studying Computer Science and I am studying Engineering.

   (b) I am not going to the movies this weekend or I am not going swimming this weekend.

4. Let $\oplus$ denote **exclusive or**, which means '… or … but not both'. The truth table for $p \oplus q$ is as follows:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

   (a) Use a truth table to show that $(p \vee q) \wedge \sim p \equiv q \wedge \sim p$.

   (b) Use a truth table to show that $(p \oplus q) \wedge r \equiv (p \wedge r) \oplus (q \wedge r)$.

[96] courses/math1061/logical-equivalence-laws.pdf

(c) Use the laws of logical equivalence and the equivalence $p \oplus q \equiv (p \vee q) \wedge \sim (p \wedge q)$ (which is the definition of $\oplus$) to show that

$$(p \oplus q) \wedge r \equiv (p \wedge r) \oplus (q \wedge r).$$

Solutions

1. (a) We have

$$\begin{aligned} \sim (\sim p \vee \sim q) &\equiv \sim (\sim p) \wedge \sim (\sim q) && \text{by De Morgan's law} \\ &\equiv p \wedge q && \text{by the double negative law (used twice).} \end{aligned}$$

(b) We have

$$\begin{aligned} \sim (p \vee \sim q) \vee (\sim p \wedge \sim q) &\equiv (\sim p \wedge \sim\sim q) \vee (\sim p \wedge \sim q) && \text{by De Morgan's law} \\ &\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) && \text{by the double negative law} \\ &\equiv \sim p \wedge (q \vee \sim q) && \text{by the distributive law} \\ &\equiv \sim p \wedge \mathbf{t} && \text{by the negation law} \\ &\equiv \sim p && \text{by the identity law.} \end{aligned}$$

2. (a) We have

p	q	r	$p \wedge q$	$q \wedge r$	$(p \wedge q) \vee (q \wedge r)$	$\sim q$	$((p \wedge q) \vee (q \wedge r)) \vee \sim q$
T	T	T	T	T	T	F	T
T	T	F	T	F	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	F	F	T	T
F	T	T	F	T	T	F	T
F	T	F	F	F	F	F	F
F	F	T	F	F	F	T	T
F	F	F	F	F	F	T	T

Thus the statement is **neither a tautology nor a contradiction**.

(b) We have

p	q	$\sim p$	$\sim p \vee q$	$\sim q$	$p \wedge \sim q$	$(\sim p \vee q) \vee (p \wedge \sim q)$
T	T	F	T	F	F	T
T	F	F	F	T	T	T
F	T	T	T	F	F	T
F	F	T	T	T	F	T

Hence the statement is a **tautology**.

3. (a) Let C be the statement "I am studying Computer Science" and E be the statement "I am studying Engineering." The given statement is $\sim (C \wedge E)$. By De Morgan's law,

$$\sim (C \wedge E) \equiv (\sim C) \vee (\sim E).$$

So an equivalent statement is: "I am not studying Computer Science or I am not studying Engineering."

(b) Let M be the statement "I am going to the movies this weekend" and S be the statement "I am going swimming this weekend." The given sentence is $(\sim M) \vee (\sim S)$. By De Morgan's law (in reverse),

$$(\sim M) \vee (\sim S) \equiv \sim (M \wedge S).$$

So an equivalent statement is: "It is not the case that I am both going to the movies this weekend and going swimming this weekend."

4. (a) We have

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$	$q \wedge \sim p$
T	T	T	F	F	F
T	F	T	F	F	F
F	T	T	T	T	T
F	F	F	T	F	F

Since the last two columns agree on every row, we have $(p \vee q) \wedge \sim p \equiv q \wedge \sim p$.

(b) We have

p	q	r	$p \oplus q$	$(p \oplus q) \wedge r$	$p \wedge r$	$q \wedge r$	$(p \wedge r) \oplus (q \wedge r)$
T	T	T	F	F	T	T	F
T	T	F	F	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

The columns for $(p \oplus q) \wedge r$ and $(p \wedge r) \oplus (q \wedge r)$ match on all rows, hence $(p \oplus q) \wedge r \equiv (p \wedge r) \oplus (q \wedge r)$.

(c) To save space, multiple laws have been applied in some steps. There are many different ways to apply the laws to prove this, so your proof may look different to this one. We have

$(p \wedge r) \oplus (q \wedge r)$	
$\equiv ((p \wedge r) \vee (q \wedge r)) \wedge \sim ((p \wedge r) \wedge (q \wedge r))$	by the given equivalence (def of $\oplus$)
$\equiv (p \vee q) \wedge r \wedge \sim ((p \wedge r) \wedge (q \wedge r))$	by distributive and commutative laws (for $\wedge r$)
$\equiv (p \vee q) \wedge r \wedge \sim ((p \wedge q) \wedge (r \wedge r))$	by associative and commutative laws (for $\wedge$)
$\equiv (p \vee q) \wedge r \wedge \sim ((p \wedge q) \wedge r)$	by idempotent law
$\equiv (p \vee q) \wedge r \wedge (\sim (p \wedge q) \vee \sim r)$	by De Morgan's law
$\equiv (p \vee q) \wedge \Big((r \wedge \sim (p \wedge q)) \vee (r \wedge \sim r)\Big)$	by distributive law (for $r \wedge$)
$\equiv (p \vee q) \wedge \Big((r \wedge \sim (p \wedge q)) \vee \mathbf{c}\Big)$	by negation law
$\equiv (p \vee q) \wedge r \wedge \sim (p \wedge q)$	by identity law
$\equiv (p \vee q) \wedge \sim (p \wedge q) \wedge r$	by associative and commutative laws (for $\wedge$)
$\equiv (p \oplus q) \wedge r$	by the given equivalence (def of $\oplus$)

Application — Applications to Computer Science

Not assessable for MATH1061. Taken from K. Rosen, Discrete Mathematics and its Applications, *7th edition, 1991, pages 20–24.*

Information is often represented using binary or **bit strings**, which are sequences consisting of zeros and ones. The **length** of a string is the total number of bits in it — for example

0110110 is a string of length 7. The logical operators $\wedge$, $\vee$ and $\oplus$ extend to binary strings. The operators AND, OR and XOR are:

p	q	$p \vee q$	$p \wedge q$	$p \oplus q$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Propositional logic also applies to the design of computer hardware. A **digital logic circuit** receives input signals (either 0, the wire being off, or 1, on) and produces output signals which are also bits. Complicated digital logic circuits can be constructed from three basic gates: the **OR gate**, the **NOT gate** and the **AND gate**.

A **combinational circuit** uses these gates and obeys the following rules.

1. Two input wires cannot be combined without a gate.
2. A single input wire can be split partway and directed into two separate gates.
3. An output wire from a gate can be used as input for another gate.
4. No output wire from a gate can eventually feed back into that gate.

For example, the logical expression $(p \wedge \sim q) \vee \sim r$ has the combinational circuit

- NOT$(q) \to n_1$
- AND$(p, n_1) \to a_1$
- NOT$(r) \to n_2$
- OR$(a_1, n_2) \to$ output $(p \wedge \sim q) \vee \sim r$

Application problems

1. Find the bitwise OR, bitwise AND and bitwise XOR of each of these pairs of binary strings.

   (a) 1011110, 0100001

   (b) 11110000, 10101010

2. Construct a combinational circuit using the NOT, OR and AND gates that produces the outputs:

   (a) $(\sim p \vee \sim r) \wedge q$

   (b) $(p \wedge \sim r) \vee (\sim q \wedge r)$

3. Find the output of the following combinational circuit, simplify the expression using logical laws and construct a new circuit for the simplified expression. The circuit is

   - NOT(p) $\to n_1$
   - NOT(q) $\to n_2$
   - OR(p, n_2) $\to o_1$
   - OR(n_1, n_2) $\to o_2$
   - AND(o_1, o_2) $\to a_1$
   - OR(a_1, r) $\to$ output

   (both p and q have their wires split, feeding two gates each).

4. The logical binary operation NAND, denoted |, is defined by

$$p \mid q \equiv \sim (p \wedge q).$$

   (a) Write down the truth table for $\sim (p \mid q)$. Show all working.

   (b) Prove, using truth tables or otherwise, that $(p \mid p) \equiv (\sim p)$. Show all working.

   (c) Prove, using truth tables or otherwise, that $((p \mid p) \mid (q \mid q)) \equiv p \vee q$. Show all working.

Application solutions

1. (a)

$$1011110 \text{ OR } 0100001 = 1111111,$$
$$1011110 \text{ AND } 0100001 = 0000000,$$
$$1011110 \text{ XOR } 0100001 = 1111111.$$

   (b)

$$11110000 \text{ OR } 10101010 = 11111010,$$
$$11110000 \text{ AND } 10101010 = 10100000,$$
$$11110000 \text{ XOR } 10101010 = 01011010.$$

2. (a) The circuit is

   - NOT(p) $\to n_1$
   - NOT(r) $\to n_2$
   - OR(n_1, n_2) $\to o_1$
   - AND(o_1, q) $\to$ output $(\sim p \vee \sim r) \wedge q$

   (b) The circuit is

- NOT$(r) \to n_1$
- AND$(p, n_1) \to a_1$
- NOT$(q) \to n_2$
- AND$(n_2, r) \to a_2$
- OR$(a_1, a_2) \to$ output $(p \wedge \sim r) \vee (\sim q \wedge r)$

3. The given digital logic circuit has output $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee r$. Using laws of logical equivalence we have

$$\begin{aligned}((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee r &\equiv (\sim q \vee (p \wedge \sim p)) \vee r && \text{by distributive and commutative laws}\\ &\equiv (\sim q \vee \mathbf{c}) \vee r && \text{by negation law}\\ &\equiv \sim q \vee r && \text{by identity law}\end{aligned}$$

The simplification is $\sim q \vee r$. The simplified logic circuit is

- NOT$(q) \to n_1$
- OR$(r, n_1) \to$ output $\sim q \vee r$

4. (a) We have

p	q	$p \wedge q$	$p \mid q$	$\sim (p \mid q)$
T	T	T	F	T
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

Hence $\sim (p \mid q)$ has the same truth values as $p \wedge q$ (it is true exactly when both p and q are true).

(b) Since $p \mid p \equiv \sim (p \wedge p)$, we have

$$p \mid p \equiv \sim (p \wedge p) \equiv \sim p$$

because $p \wedge p \equiv p$ (idempotent law). Alternatively, by truth table:

p	$p \wedge p$	$p \mid p$	$\sim p$
T	T	F	F
F	F	T	T

The last two columns agree, so $p \mid p \equiv \sim p$.

(c) First note from part (b) that $p \mid p \equiv \sim p$ and $q \mid q \equiv \sim q$. Therefore

$$\begin{aligned}(p \mid p) \mid (q \mid q) &\equiv (\sim p) \mid (\sim q) && \text{by part (b)}\\ &\equiv \sim((\sim p) \wedge (\sim q)) && \text{by def of NAND}\\ &\equiv p \vee q && \text{by De Morgan's law.}\end{aligned}$$

Alternatively, by truth table:

p	q	$p \mid p$	$q \mid q$	$(p \mid p) \mid (q \mid q)$	$p \vee q$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

The last two columns match, hence $(p \mid p) \mid (q \mid q) \equiv p \vee q$.

§3 — Conditional Statements

Epp 4th ed. pp. 039–050 · 5th ed. (metric) pp. 053–066. Lecture 3 (week 1). Notes: conditional-statements[97]*.*

Problems

1. Let p, q and r be statements. Use the laws of logical equivalence to show that

$$p \to (q \vee r) \equiv (p \wedge \sim q) \to r.$$

2. Let p and q be statement variables. Are the following statements logically equivalent? Justify your answer.

   - **Statement A:** $(p \to q) \wedge \sim q$
   - **Statement B:** $\sim (p \wedge q)$

3. Write each of the following statements in the form 'if … then …'.

   (a) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.

[97] courses/math1061/conditional-statements.pdf

(b) Jane gets seasick whenever she is on a boat.

4. For each of the following, write down a truth table for the statement, and determine whether the statement is a tautology, a contradiction, or neither.

   (a) $(\sim p \wedge (p \to q)) \to \sim q$

   (b) $(p \to (q \vee r)) \leftrightarrow ((p \wedge \sim q) \to r)$

5. Negate the following two statements.

   (a) If it rains, then Sue takes her umbrella.

   (b) The cakes burn if the oven temperature is too high.

6. Recall "a sufficient condition for s is r" means r is a sufficient condition for s, and that "a necessary condition for s is r" means r is a necessary condition for s. Rewrite the following statements in 'if … then …' form:

   (a) A sufficient condition for John's team to win the championship is that it wins the rest of its games.

   (b) A necessary condition for this computer program to be correct is that it not produce error messages during translation.

Solutions

1. We have

$$\begin{aligned} p \to (q \vee r) &\equiv \sim p \vee (q \vee r) && \text{by definition of implication} \\ &\equiv (\sim p \vee q) \vee r && \text{by the associative law} \\ &\equiv \sim (p \wedge \sim q) \vee r && \text{by De Morgan's law and double negative law} \\ &\equiv (p \wedge \sim q) \to r && \text{by definition of implication.} \end{aligned}$$

2. Let A be the statement $(p \to q) \wedge \sim q$ and let B be the statement $\sim (p \wedge q)$. Then

$$\begin{aligned} A &\equiv (\sim p \vee q) \wedge \sim q && \text{by definition of implication} \\ &\equiv (\sim p \wedge \sim q) \vee (q \wedge \sim q) && \text{by distributive and commutative laws} \\ &\equiv (\sim p \wedge \sim q) \vee \mathbf{c} && \text{by negation law} \\ &\equiv \sim p \wedge \sim q && \text{by identity law,} \end{aligned}$$

   while

$$B \equiv \sim (p \wedge q) \equiv \sim p \vee \sim q \qquad \text{by De Morgan's law.}$$

Thus Statement A and Statement B are **not** logically equivalent because, for example, if p is false and q is true, then A is false and B is true. Hence $A \not\equiv B$.

3. (a) If you bought the computer less than a year ago, then the warranty is good.

   (b) If Jane is on a boat, then Jane gets seasick.

4. (a) **Solution Option 1.** The truth table for the statement is

p	q	$\sim p$	$p \to q$	$\sim p \wedge (p \to q)$	$\sim q$	$(\sim p \wedge (p \to q)) \to \sim q$
T	T	F	T	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

The statement is **neither a tautology nor a contradiction**.

Solution Option 2. The same truth table, written under the parts of the statement:

p	q	$\sim p$	$\wedge$	$(p \to q)$	$\to$	$\sim q$
T	T	F	F	T	T	F
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T
order						

The small numbers show the order in which the columns were completed. Column comes from comparing columns and . Column comes from comparing columns and , and shows the truth values for the entire statement. This shows that this statement is neither a tautology nor a contradiction. (You don't have to number the columns — this is just to help your understanding!)

(b) **Solution Option 1.** The truth tables for the LHS and RHS are:

p	q	r	$q \vee r$	$p \rightarrow (q \vee r)$	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \rightarrow r$
T	T	T	T	T	F	F	T
T	T	F	T	T	F	F	T
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	F	T
F	F	F	F	T	T	F	T

Therefore the truth table for the statement is:

p	q	r	$(p \rightarrow (q \vee r)) \leftrightarrow ((p \wedge \sim q) \rightarrow r)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

The statement is a **tautology**.

Solution Option 2. Another way to represent the truth table for $(p \rightarrow (q \vee r)) \leftrightarrow ((p \wedge \sim q) \rightarrow r)$:

p	q	r	$p \rightarrow$	$(q \vee r)$	$\leftrightarrow$	$(p \wedge \sim q)$	$\rightarrow r$
T	T	T	T	T	T	F	T
T	T	F	T	T	T	F	T
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	T	F	T
F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	F	T	F	T
order							

Here column comes from comparing columns and . We see (from column) that this is a tautology.

5. (a) Let R be the statement "it rains" and U be the statement "Sue takes her umbrella." The given statement is $R \to U$. Its negation is

$$R \wedge \sim U,$$

i.e. "It rains and Sue does not take her umbrella."

(b) Let H be the statement "the oven temperature is too high" and B be the statement "the cakes burn." The given statement is $H \to B$. Its negation is

$$H \wedge \sim B,$$

i.e. "The oven temperature is too high and the cakes do not burn."

6. (a) If John's team wins the rest of its games, then John's team will win the championship.

(b) If this computer program is correct, then it does not produce error messages during translation.

§4 — Valid and Invalid Arguments

Epp 4th ed. pp. 051–063 · 5th ed. (metric) pp. 066–079. Lecture 4 (week 2). Notes: valid-argument-forms[98]*.*

Problems

1. State truth values for the statement variables p and q that demonstrate that the following argument is invalid, and justify your answer.

   Premise 1: q Premise 2: $p \to q$ Conclusion: p

2. Find values of the statement variables a, b and c that show that the following argument is invalid.

   1. $a \to b$
   2. $\sim a \to c$
   3. $\sim b \to c \therefore b$

3. Use a truth table to show that the following argument forms are valid.

[98] courses/math1061/valid-argument-forms.pdf

(a) **Elimination:**

1. $p \vee q$
2. $\sim q \therefore p$

(b) **Generalisation:**

1. $p \therefore p \vee q$

4. Use a truth table to show that the following argument is invalid.

1. $(p \wedge q) \rightarrow \sim r$
2. $p \vee \sim q$
3. $\sim q \rightarrow p \therefore \sim r$

Solutions

1. To show the argument is invalid, we must find truth values for p and q such that the premises are true and the conclusion is false.

   Suppose p is false and q is true. Then Premise 1 (q) is true and Premise 2 ($p \rightarrow q$) is true (by definition of implication). However, the conclusion (p) is false. Therefore the argument is invalid.

2. To show the argument is invalid, we must find truth values for a, b and c such that the premises are true and the conclusion is false.

   Let a be false, b be false, and c be true. By definition of implication, all three premises are true, but the conclusion (b) is false, so the argument is invalid.

3. (a)

p	q	premise 1: $p \vee q$	premise 2: $\sim q$	conclusion: p
T	T	T	F	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

   There is only one row of the truth table in which all premises are true (the second row), and here the conclusion is true. Thus, whenever all the premises are true, the conclusion is true, so the argument is **valid**.

   (b)

p	q	premise 1: p	conclusion: $p \vee q$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	F	F

There are two rows for which all premises (here just the one premise) are true — the first two rows — and for each of these rows the conclusion is true. Thus, whenever all the premises are true, the conclusion is true, so the argument is **valid**.

4. We compute the truth table:

p	q	r	$(p \wedge q) \rightarrow \sim r$	$p \vee \sim q$	$\sim q \rightarrow p$	Prem. 1 $\wedge$ Prem. 2 $\wedge$ Prem. 3	$\sim r$
T	T	T	F	T	T	F	F
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	F
T	F	F	T	T	T	T	T
F	T	T	T	F	T	F	F
F	T	F	T	F	T	F	T
F	F	T	T	T	F	F	F
F	F	F	T	T	F	F	T

There are three rows for which all the premises are true, however for one of these rows the conclusion is false. In particular, taking p to be true, q to be false, and r to be true makes all the premises true but the conclusion false, so the argument is **invalid**.

§5 — Methods for Determining Validity

Epp 4th ed. pp. 051–063 · 5th ed. (metric) pp. 066–079. Lecture 5 (week 2). Notes: determining-argument-validity[99].

Problems

1. Determine whether the following arguments are valid. If the argument is valid, prove it using rules of inference. If the argument is invalid, demonstrate it with a choice of truth values for the variables.

[99] courses/math1061/determining-argument-validity.pdf

(a)

$p \to q \; q \to p \therefore p \vee q$

(b)

$p \; p \to q \sim q \vee r \therefore r$

(c)

$p \vee q \; p \to \sim q \; p \to r \therefore r$

2. Write the following argument in symbolic form. Then determine whether the argument is valid or not.

   > An error message is received if the output is flagged. The system is running normally only if the output is not flagged. An error message is received. Therefore, the system is not running normally.

   Please use the following variables:

p	the output is flagged
q	an error message is received
r	the system is running normally

3. Give truth values for the variables p, q, r, s and w which demonstrate that the following argument is invalid.

$$[(p \vee q) \wedge (q \to r) \wedge ((p \wedge s) \to w) \wedge (\sim r) \wedge (q \to s)] \to w$$

Solutions

1. (a) The argument is **invalid**. When p and q are both false, all the premises are true, but the conclusion is false. One could arrive at this example by computing the following truth table:

p	q	$p \to q$	$q \to p$	$p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	F

   (In the last row, the premises are true, but the conclusion is false.)

(b) The argument is **valid**, and we prove this with rules of inference:

1.	p	Premise
2.	$p \to q$	Premise
3.	$\sim q \vee r$	Premise
4.	q	from 1 and 2 by Modus Ponens
5.	r	from 3 and 4 by elimination (and the double negative law)

Therefore the argument is valid.

(c) Suppose the argument is invalid. Then it is possible to find truth values for the statement variables that make all the premises true and the conclusion false. For the conclusion to be false, we have r **is false**. Since r is false but the third premise $(p \to r)$ is true, we must have p **is false**. Since p is false, this means the second premise $(p \to \sim q)$ is also true. Now since p is false but the first premise $(p \vee q)$ is true, we must have q **is true**.

Thus, with these truth values for the three variables, we have that all the premises are true, but the conclusion is false. So the argument is **invalid**.

2. The argument has the symbolic form:

1.	$p \to q$	(An error message is received if the output is flagged.)
2.	$r \to \sim p$	(The system is running normally only if the output is not flagged.)
3.	q	(An error message is received.)
$\therefore$	$\sim r$	(The system is not running normally.)

Suppose the argument is invalid. Then it is possible to find truth values for the statement variables that make all the premises true and the conclusion false. For the conclusion to be false, we have r **is true**. Since r is true, and premise 2 is true, we must have $\sim p$ true, that is p **is false**. Since premise 3 is true, we have q **is true**.

Thus, with these truth values for the three variables, we have that all the premises are true, but the conclusion is false. So the argument is **invalid**.

3. The argument being valid is the same as this overall implication statement being a tautology.

$$\big[(p \vee q) \wedge (q \to r) \wedge ((p \wedge s) \to w) \wedge (\sim r) \wedge (q \to s)\big] \to w$$

We can write the argument as follows:

1. $p \vee q$
2. $q \to r$
3. $(p \wedge s) \to w$
4. $\sim r$
5. $q \to s \therefore w$

Suppose the argument is invalid. Then it is possible to find truth values for the statement variables that make all the premises true and the conclusion false. For the conclusion to be false, we have w **is false**. Since premise 4 is true we have r **is false**. Since premise 2 is true and r is false, we must have q **is false**. Since premise 1 is true and q is false we have p **is true**. Since premise 3 is true and w is false, we must have $(p \wedge s)$ false, and since p is true, this means s **is false**. These truth values also mean premise 5 is true.

Thus with these truth values for the variables, we have that all the premises are true, but the conclusion is false. So the argument is **invalid**.

Application — Applications of Valid and Invalid Arguments

Not assessable for MATH1061. Taken from K. Rosen, Discrete Mathematics and its Applications, *7th edition, 1991, pages 48–68.*

Application problems

1. Translating English sentences into logical expressions is an essential part of specifying both hardware and software systems. Software engineers must take requirements in English and produce precise, unambiguous specifications that can be used as the basis for system development.

   Consider the following predicates and domains:

   Predicates:

$F(x, y)$	disk x has more than y kilobytes of free space.
$S(m)$	mail message m can be saved.
$A(x)$	alert x is active.
$Q(m)$	message m is queued.
$T(m)$	message m is transmitted.

$D(s)$	the diagnostic monitor tracks the status of system s.

Domains:

M	the nonempty set of all messages.
X	the set of all disks.
Y	the set of all alerts.
S	the set of all systems.

Express the system specifications below using quantifiers and logical connectives:

(a) At least one mail message, among the nonempty set of messages, can be saved if there is a disk with more than 10 kilobytes of free space.

(b) Whenever there is an active alert, all queued messages are transmitted.

(c) The diagnostic monitor tracks the status of all systems except the main console.

2. System specifications should be **consistent**; that is, they should not contain conflicting requirements that could be used to derive a contradiction. When specifications are not consistent, there would be no way to develop a system that satisfies all specifications. Logically, a system is consistent if it is possible for all the premises to be simultaneously true.

   Use logical expressions and methods of logical deduction to determine whether the following specifications are consistent.

   (a) The system is in multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode.

   (b) The router can send packets to the edge system only if it supports the new address space. For the router to support the new address space it is necessary that the latest software release be installed. The router can send packets to the edge system if the latest software release is installed. The router does not support the new address space.

Application solutions

1. (a) $(\exists x \in X \text{ s.t. } F(x, 10)) \to (\exists m \in M \text{ s.t. } S(m))$.

   (b) $(\exists y \in Y \text{ s.t. } A(y)) \to (\forall m \in M,\ (Q(m) \to T(m)))$.

   (c) Let $c \in S$ denote the main console. $\forall s \in S,\ (s \neq c \to D(s))$.

2. (a) Define the following variables:

m	the system is in multiuser state
n	the system is operating normally
k	the kernel is functioning
i	the system is in interrupt mode

   Now we can express the specifications as follows:

   1. $m \leftrightarrow n$
   2. $n \to k$
   3. $\sim k \vee i$
   4. $\sim m \to i$
   5. $\sim i$

   Solution Option 1. If these specifications are consistent, then we must be able to find truth values for each of the variables that makes all of the specifications true. Assume the specifications are consistent (so we assume each specification is true). For specification 5 to be true we know i **is false**. For specification 4 to be true, with i false, we know $\sim m$ is false so m **is true**. For premise 1 to be true, with m true, we have n **is true**. For specification 2 to be true, and since n is true, we have k **is true**. However, now $\sim k$ is false and i is false, which means specification 3 is false. This contradicts our assumption that all the specifications are true, and so it is impossible for the system to be consistent. Therefore, the system is **not consistent**.

   Solution Option 2. Suppose the specifications are all true. Using laws of inference, we have

1.	$m \leftrightarrow n$	specification
2.	$n \to k$	specification
3.	$\sim k \vee i$	specification
4.	$\sim m \to i$	specification
5.	$\sim i$	specification
6.	$\sim k$	from 3 and 5 by elimination (and double negative law)
7.	$\sim n$	from 2 and 6 by Modus Tollens

8.	$(m \to n) \wedge (n \to m)$	from 1 by definition of $\leftrightarrow$
9.	$m \to n$	from 8 by specialisation
10.	$\sim m$	from 9 and 7 by Modus Tollens
11.	i	from 4 and 10 by Modus Ponens

But now we have both i and $\sim i$ being true, which is a contradiction. This contradicts our assumption that all the specifications are true, and so it is impossible for the system to be consistent. Therefore, the system is **not consistent**.

(b) Define the following variables:

r	the router can send packets to the edge system
a	the router supports the new address space
s	the latest software release is installed

Now we can express the specifications as follows:

1. $r \to a$
2. $a \to s$
3. $s \to r$
4. $\sim a$

Assume the specifications are consistent, so each specification is true. For specification 4 to be true, $\sim a$ is true, so a **is false**. For specification 1 to be true with a false we must have r **is false**. Now, since r is false and specification 3 is true, we have s **is false**. We double-check that specification 2 is true, which it is because a is false. Thus, we have found truth values of the statement variables that make all of the specifications true, so the system **is consistent**.

§6 — Quantified Statements

Epp 4th ed. pp. 096–108 · 5th ed. (metric) pp. 108–121. Lecture 5 (week 2). Notes: quantified-statements[100].

[100] courses/math1061/quantified-statements.pdf

Problems

1. Rewrite each of the following in the form "$\forall$__, __." Here, let D be the set of all dinosaurs, and let L be the set of all logicians.

   (a) All dinosaurs are extinct.

   (b) Every real number is positive, negative, or zero.

   (c) No irrational numbers are integers.

   (d) No logicians are lazy.

   (e) The number 2,147,581,953 is not equal to the square of any integer.

   (f) The number -1 is not equal to the square of any real number.

2. Rewrite each of the following in the form "$\exists$__ such that __." Here, let X be the set of all exercises.

   (a) Some exercises have answers.

   (b) Some real numbers are rational.

3. Determine whether the following statements are true or false. Justify your reasoning.

   (a) $\forall x \in \mathbb{R},\ x \geq \frac{1}{x}$.

   (b) $\forall a \in \mathbb{Z},\ \frac{a-1}{a} \notin \mathbb{Z}$.

   (c) $\forall m, n \in \mathbb{Z}^+,\ mn \geq m + n$.

Solutions

1. (a) $\forall x \in D,\ x$ is extinct.

   (b) $\forall x \in \mathbb{R},\ (x > 0 \vee x < 0 \vee x = 0)$.

   (c) $\forall x \in \mathbb{R},\ (x \text{ is irrational} \rightarrow x \notin \mathbb{Z})$.

   (d) $\forall x \in L,\ x$ is not lazy.

   (e) $\forall x \in \mathbb{Z},\ x^2 \neq 2147581953$.

   (f) $\forall x \in \mathbb{R},\ -1 \neq x^2$.

2. (a) $\exists x \in X$ such that x has an answer.

   (b) $\exists x \in \mathbb{R}$ such that $x \in \mathbb{Q}$.

3. (a) **False.** Counterexample: take $x = \frac{1}{2}$. Then $x = \frac{1}{2}$ but $\frac{1}{x} = 2$, so $x \geq \frac{1}{x}$ is false.

(b) **False.** Counterexample: take $a = 1$. Then $\frac{a-1}{a} = \frac{0}{1} = 0 \in \mathbb{Z}$.

(c) **False.** Counterexample: take $n = m = 1$. Then $mn = 1$ but $m + n = 2$.

§7 — Negation of Quantified Statements

Epp 4th ed. pp. 108–117 · 5th ed. (metric) pp. 122–131. Lecture 6 (week 2). Notes: quantified-statements[101].

Problems

1. Write the negation of each of the following statements, and then determine whether the original or the negation is true:

   (a) $\forall x \in \mathbb{R}$, if $x^2 > 9$ then $x \geq 2$.

   (b) $\forall n \in \mathbb{Z}^+$, n is either prime or composite.

   (c) $\exists x, y \in \mathbb{R}$ such that $x \notin \mathbb{Z}$ and $y \notin \mathbb{Z}$ but $xy \in \mathbb{Z}$.

2. Write down the negation of each of the following statements:

   (a) All fire engines are red.

   (b) Some apples are ripe.

   (c) A triangle with equal sides exists.

Solutions

1. (a) **Original:** $\forall x \in \mathbb{R}$, if $x^2 > 9$ then $x \geq 2$. **Negation:** $\exists x \in \mathbb{R}$ such that $x^2 > 9$ and $x < 2$.

   The original statement is **false**. For instance, take $x = -4$. Then $x^2 > 9$ but $x \not\geq 2$.

   (b) **Original:** $\forall n \in \mathbb{Z}^+$, n is either prime or composite. **Negation:** $\exists n \in \mathbb{Z}^+$ such that n is neither prime nor composite.

   The original statement is **false**. For instance, consider $n = 1$, which is neither prime nor composite.

   (c) **Original:** $\exists x, y \in \mathbb{R}$ such that $x \notin \mathbb{Z}$ and $y \notin \mathbb{Z}$ but $xy \in \mathbb{Z}$. **Negation:** $\forall x, y \in \mathbb{R}$, if $x \notin \mathbb{Z}$ and $y \notin \mathbb{Z}$ then $xy \notin \mathbb{Z}$.

[101] courses/math1061/quantified-statements.pdf

The original statement is **true**. For instance, take $x = \sqrt{2}$ and $y = \sqrt{2}$. Then $x \notin \mathbb{Z}$ and $y \notin \mathbb{Z}$, but $xy = 2 \in \mathbb{Z}$.

2. (a) **Original:** All fire engines are red. **Negation:** There exists a fire engine which is not red.

   (b) **Original:** Some apples are ripe. **Negation:** All apples are not ripe.

   (c) **Original:** A triangle with equal sides exists. **Negation:** There does not exist a triangle with equal sides.

§8 — Statements with Multiple Quantifiers

Epp 4th ed. pp. 117–131 · 5th ed. (metric) pp. 131–146. Lecture 6 (week 2). Notes: quantified-statements[102].

Problems

1. Write down the negation of each of the following statements. Then determine whether the statement is true, or whether its negation is true.

   (a) $\forall x, y \in \mathbb{Z}$, $\exists z \in \mathbb{Z}$ such that $z = x - y$.

   (b) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x^2 - xy - 2y^2 = 0$.

   (c) $\exists x \in \mathbb{Z}^+$ such that $\forall y \in \mathbb{Z}^+$, $x \geq y$.

   (d) $\forall r, s \in \mathbb{R}$, if $rs \in \mathbb{Q}$, then $r \in \mathbb{Q}$ and $s \in \mathbb{Q}$.

2. For each of the following English sentences:

   (a) For any rational number x and any negative integer y there exists a real number that is greater than the product of x and y.

   (b) For every real number x, there is a real number y for which $3y = x$.

   do the following:

   i. Write the statement in symbolic form.

   ii. Write the negation of the statement in symbolic form.

   iii. Determine which of the statement or its negation is true.

[102] courses/math1061/quantified-statements.pdf

Solutions

1. (a) **Original:** $\forall x, y \in \mathbb{Z}$, $\exists z \in \mathbb{Z}$ such that $z = x - y$. **Negation:** $\exists x \in \mathbb{Z}$ $\exists y \in \mathbb{Z}$ such that $\forall z \in \mathbb{Z}$, $z \neq x - y$.

   The original statement is **true** (and the negation is false). For any given $x, y \in \mathbb{Z}$, choose $z = x - y \in \mathbb{Z}$.

   (b) **Original:** $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x^2 - xy - 2y^2 = 0$. **Negation:** $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$, $x^2 - xy - 2y^2 \neq 0$.

   The original statement is **true** (and the negation is false). For any $x \in \mathbb{R}$, taking $y = \frac{x}{2}$ gives

$$x^2 - x\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)^2 = x^2 - \frac{x^2}{2} - 2 \cdot \frac{x^2}{4} = x^2 - \frac{x^2}{2} - \frac{x^2}{2} = 0.$$

   (c) **Original:** $\exists x \in \mathbb{Z}^+$ such that $\forall y \in \mathbb{Z}^+$, $x \geq y$. **Negation:** $\forall x \in \mathbb{Z}^+$ $\exists y \in \mathbb{Z}^+$ such that $x < y$.

   The **negation** is true (and the original is false). For any $x \in \mathbb{Z}^+$, take $y = x + 1$. Then y is an integer satisfying the inequality $x < y$.

   (d) **Original:** $\forall r, s \in \mathbb{R}$, if $rs \in \mathbb{Q}$, then $r \in \mathbb{Q}$ and $s \in \mathbb{Q}$. **Negation:** $\exists r, s \in \mathbb{R}$ such that $rs \in \mathbb{Q}$ and ($r \notin \mathbb{Q}$ or $s \notin \mathbb{Q}$).

   The **negation** is true (and the original is false). Take $r = \sqrt{2}$ and $s = \sqrt{2}$. Then $rs = 2 \in \mathbb{Q}$ but $r, s \notin \mathbb{Q}$.

2. (a) i. **Original:** $\forall x \in \mathbb{Q}$, $\forall y \in \mathbb{Z}^{<0}$, $\exists z \in \mathbb{R}$ s.t. $z > xy$.

   ii. **Negation:** $\exists x \in \mathbb{Q}$ $\exists y \in \mathbb{Z}^{<0}$ s.t. $\forall z \in \mathbb{R}$, $z \leq xy$.

   iii. The **original** statement is true. For any given $x \in \mathbb{Q}$ and $y \in \mathbb{Z}^{<0}$, take $z = xy + 1$. This is a real number satisfying the inequality $z > xy$.

   (b) i. **Original:** $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ s.t. $3y = x$.

   ii. **Negation:** $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$, $3y \neq x$.

   iii. The **original** statement is true. Given any $x \in \mathbb{R}$, choose $y = \frac{x}{3}$. This is a real number satisfying the equality $3y = x$.

§9 — Direct Proofs and Counterexamples

Epp 4th ed. pp. 145–163 · 5th ed. (metric) pp. 160–182. Lecture 7 (week 3). Notes: proof-techniques[103].

Problems

1. For each of the following statements, either prove the statement or disprove it using a counterexample.

   (a) The sum of any pair of even integers is even.

   (b) There exist positive integers a, b and c such that $a^2 + b^2 = c^2$.

   (c) There is an even integer n such that $5n - 4$ is prime.

   (d) $\forall n \in \mathbb{Z}^+$, if $n \geq 4$ then $2n^2 - 5n + 2$ is composite.

   (e) $\forall x, y \in \mathbb{R}$, if $y^2 > x^2$ then $y > x$.

2. Find a counterexample that disproves the following statement:

   $$\text{For all } x \in \mathbb{R},\ \lfloor x^2 \rfloor = \lfloor x \rfloor^2.$$

   (Recall that $\lfloor \cdot \rfloor$ denotes the floor function.)

3. Find the errors in the following proofs and rewrite the proof correctly. Here is a list of common errors made when writing proofs:

   - Arguing from examples
   - Using the same variable to mean two different things
   - Jumping to the conclusion
   - Assuming what is to be proved
   - Confusion between what is known and what is to be shown
   - Use of the word *any* instead of *some*
   - Misuse of the word *if*

   (a) The difference between any odd integer and any even integer is odd.

   Proof. Let m be any odd integer, then $m = 2k + 1$ where $k \in \mathbb{Z}$ and let n be any even integer, so $n = 2k$ for any $k \in \mathbb{Z}$. Then $m - n = 2k + 1 - 2k = 1$ and 1 is odd so the statement is true.

   (b) If a and b are even integers, then $4a + 2b$ is even.

[103] courses/math1061/proof-techniques.pdf

Proof. We have $a = 2k$ and $b = 2j$ for some $k, j \in \mathbb{Z}$. Then we want to show that $4a + 2b$ is even, so

$$\begin{aligned} 4a + 2b &= 2\ell \\ 4(2k) + 2(2j) &= 2\ell \\ 8k + 4j &= 2\ell \\ 2(4k + 2j) &= 2\ell \end{aligned}$$

therefore the statement is true.

(c) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $xy = 1$.

Proof. This is true, for example if $x = 10$ then let $y = \frac{1}{10}$ and $xy = 10 \cdot \frac{1}{10} = 1$. Therefore the statement is true.

(d) $\forall n \in \mathbb{Z}$, if $n > 0$ then $n^2 + 2n + 1$ is composite.

Proof. Suppose n is an integer and $n > 0$, then we want to show that n^2+2n+1 is composite, so this means

$$n^2 + 2n + 1 = rs$$

where

$$1 < r, s < n^2 + 2n + 1$$

Clearly neither r nor s are equal to 1 so $n^2 + 2n + 1$ is not prime and hence must be composite.

Solutions

1. (a) **The sum of any pair of even integers is even.**

Proof. Let $m, n \in \mathbb{Z}$ be even. Then $m = 2k$ and $n = 2\ell$ for some $k, \ell \in \mathbb{Z}$. Hence

$$m + n = 2k + 2\ell = 2(k + \ell),$$

which is even. Therefore, the sum of any pair of even integers is even. □

(b) **There exist positive integers a, b and c such that $a^2 + b^2 = c^2$.**

Proof. Consider the positive integers $a = 3$, $b = 4$, $c = 5$. Then $a^2 + b^2 = 9 + 16 = 25 = c^2$. □

(c) **There is an even integer n such that $5n - 4$ is prime.** This statement is **false**. We prove the negation: $\forall n \in \mathbb{Z}^{\text{even}}$, $5n - 4$ is not prime.

Proof. Suppose that n is an even integer. Thus $n = 2k$ for some integer k. Hence

$$5n - 4 = 5(2k) - 4 = 10k - 4 = 2(5k - 2).$$

By definition, this means that $5n - 4$ is even. The only even prime is 2. If $2(5k - 2) = 2$, then $5k - 2 = 1$ and hence $k = \frac{3}{5}$ which is not an integer. Therefore $5n - 4$ is not a prime number. □

(d) $\forall n \in \mathbb{Z}^+$**, if** $n \geq 4$ **then** $2n^2 - 5n + 2$ **is composite.**

Proof. Suppose n is a positive integer and $n \geq 4$. Let us factor

$$2n^2 - 5n + 2 = (2n - 1)(n - 2).$$

Since $n \geq 4$, we have $2n - 1 \geq 7$ and $n - 2 \geq 2$, so both factors are integers > 1. Hence the expression is composite for all $n \geq 4$. □

(e) $\forall x, y \in \mathbb{R}$**, if** $y^2 > x^2$ **then** $y > x$**.** This statement is **false**. Take $x = 1$ and $y = -2$. Then $y^2 = 4 > x^2 = 1$, but $y = -2 \not> 1 = x$.

2. Take $x = \frac{3}{2}$. Then $\lfloor x^2 \rfloor = \lfloor \frac{9}{4} \rfloor = 2$, while $\lfloor x \rfloor^2 = \lfloor \frac{3}{2} \rfloor^2 = 1^2 = 1$. Hence $\lfloor x^2 \rfloor \neq \lfloor x \rfloor^2$, so the statement is false.

3. (a) **Statement.** The difference between any odd integer and any even integer is odd.

Proof. Let m be any odd integer, then $m = 2k + 1$ where $k \in \mathbb{Z}$ and let n be any even integer, so $n = 2k$ for any $k \in \mathbb{Z}$. [**Error: using** k **for both** m **and** n **means that the statement is only proved when** $m = n + 1$**.**] Then $m - n = 2k + 1 - 2k = 1$ and 1 is odd so the statement is true.

Correct proof. Let m be any odd integer and let n be any even integer. Then there exist $k, \ell \in \mathbb{Z}$ such that $m = 2k + 1$ and $n = 2\ell$. Hence

$$m - n = (2k + 1) - 2\ell = 2(k - \ell) + 1,$$

which is an odd integer by definition. □

(b) **Statement.** If a and b are even integers, then $4a + 2b$ is even. [**Error: the proof forgets to state its assumptions.**]

Proof. We have $a = 2k$ and $b = 2j$ for some $k, j \in \mathbb{Z}$. Then we want to show that $4a + 2b$ is even, so

$$\begin{aligned} 4a + 2b &= 2\ell \quad \text{[Error: this line assumes what is to be proved.]} \\ 4(2k) + 2(2j) &= 2\ell \\ 8k + 4j &= 2\ell \\ 2(4k + 2j) &= 2\ell \end{aligned}$$

therefore the statement is true.

Correct proof. Suppose a and b are even integers. Then $a = 2k$ and $b = 2j$ for some $k, j \in \mathbb{Z}$. Therefore

$$4a + 2b = 4(2k) + 2(2j) = 8k + 4j = 2(4k + 2j).$$

Since $k, j \in \mathbb{Z}$, we have $(4k + 2j) \in \mathbb{Z}$ and so, by definition, $4a + 2b$ is even. □

(c) **Statement.** $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $xy = 1$.

Proof. This is true, for example if $x = 10$ then let $y = \frac{1}{10}$ and $xy = 10 \cdot \frac{1}{10} = 1$. Therefore the statement is true. **[Error: the proof argues from a single example, which cannot establish a universal statement. Moreover, the statement is false.]**

Correct disproof. The statement is false. As a counterexample, let $x = 0$. We know that $0 \cdot y = 0$ for all $y \in \mathbb{R}$ and so there is no real number y such that $0 \cdot y = 1$. □

(d) **Statement.** $\forall n \in \mathbb{Z}$, if $n > 0$ then $n^2 + 2n + 1$ is composite.

Proof. Suppose n is an integer and $n > 0$, then we want to show that n^2+2n+1 is composite, so this means

$n^2+2n+1 = rs$ **[Error: forgets to define r, s — e.g. for some $r, s \in \mathbb{Z}$]**

where

$$1 < r, s < n^2 + 2n + 1$$

Clearly neither r nor s are equal to 1 so $n^2 + 2n + 1$ is not prime and hence must be composite. **[Error: this is jumping to the conclusion without showing what needs to be shown. The expression we need to show (the definition of composite) has been given but it has not been shown.]**

Correct proof. Let $n \in \mathbb{Z}$ with $n > 0$. Then we have

$$n^2 + 2n + 1 = (n + 1)(n + 1).$$

Now since $0 < n < n^2 + 2n$, we must have that $1 < n + 1 < n^2 + 2n + 1$. So by letting $r = s = n + 1$ we have shown that

$$n^2 + 2n + 1 = rs$$

for integers $1 < r, s < n^2 + 2n + 1$. By definition this means $n^2 + 2n + 1$ is composite. □

Application — Applications of Proofs to Computer Science

Not assessable for MATH1061. Taken from K. Rosen, Discrete Mathematics and its Applications, *7th edition, 1991, pages 74–80.*

Computer programs have been developed to automate the task of reasoning and proving theorems. Many of these programs make use of a rule of inference known as **resolution**. This rule of inference is based on the tautology

$$((p \vee q) \wedge (\sim p \vee r)) \rightarrow (q \vee r) \tag{1}$$

Resolution can be used to build automatic theorem proving systems.

To construct proofs using resolution as the only rule of inference, the hypotheses and conclusion must be expressed as **clauses**. Clauses are logical expressions which only feature disjunctions of variables or their negations (i.e. expressions which only use $\vee$, $\sim$, so they do not use $\wedge$, $\rightarrow$). To express any statement as a clause, we can use the laws of logical equivalence.

Application problems

1. (a) Using a truth table, show that expression (1) is a tautology.

   (b) Write the following expressions as clauses.

   i. $\sim (\sim r \wedge w)$

   ii. $r \rightarrow u$

   iii. $\sim (u \wedge w)$

   (c) Use resolution only to show that the following argument is valid:

   If it is raining, then Yvette has their umbrella. It is not the case that Yvette has their umbrella and gets wet. It is not the case that Yvette gets wet while it is not raining. Therefore, Yvette does not get wet.

Application solutions

1. (a) The truth table is

p	q	r	$p \vee q$	$\sim p \vee r$	$(p \vee q) \wedge (\sim p \vee r)$	$q \vee r$	$\big((p \vee q) \wedge (\sim p \vee r)\big) \rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	F	T

Since the last column is always T, the statement is a tautology.

(b) Clause forms:

i. $\sim (\sim r \wedge w) \equiv r \vee \sim w$ (by De Morgan's law and double negative law).

ii. $r \rightarrow u \equiv \sim r \vee u$ (by definition of implication).

iii. $\sim (u \wedge w) \equiv \sim u \vee \sim w$ (by De Morgan's law).

(c) Define the following variables:

r	it is raining
u	Yvette has their umbrella
w	Yvette gets wet

Then we can express the argument as

1. $r \rightarrow u$
2. $\sim (u \wedge w)$
3. $\sim (\sim r \wedge w) \therefore \sim w$

Using the equivalences in (b), we can rewrite this as

1.	$\sim r \vee u$	
2.	$\sim u \vee \sim w$	
3.	$r \vee \sim w$	
4.	$\sim r \vee \sim w$	From 1 and 2 by Resolution
5.	$\sim w$	From 3 and 4 by Resolution.

§10 — Proof by Contradiction

Epp 4th ed. pp. 198–207 · 5th ed. (metric) pp. 218–228. Lecture 8 (week 3). Notes: proof-techniques[104].

Problems

1. Prove the following statement using a proof by contradiction:

$$\text{For integers } a \text{ and } b, \text{ if } 6a + 3b \text{ is odd, then } b \text{ is odd.}$$

2. Use a proof by contradiction to prove the following statement:

$$\text{For all positive integers } x \text{ and } y, \ x^2 - y^2 \neq 1.$$

Solutions

1. **For integers a and b, if $6a + 3b$ is odd, then b is odd.**

   Proof. Suppose the statement is false, that is, suppose that there exist integers a and b for which $6a + 3b$ is odd and b is even.

   Since b is even, $b = 2k$ for some integer k. Thus

   $$6a + 3b = 6a + 3(2k) = 6a + 6k = 2(3a + 3k).$$

   Since $a, k \in \mathbb{Z}$, $3a + 3k \in \mathbb{Z}$ so $6a + 3b$ is even. This contradicts our assumption that $6a + 3b$ is odd, so our assumption was incorrect and hence the original statement is true. □

2. **For all positive integers x and y, $x^2 - y^2 \neq 1$.**

   Proof. Suppose, for contradiction, that x and y are positive integers satisfying $x^2 - y^2 = 1$. Now, by factoring, we have

   $$1 = x^2 - y^2 = (x - y)(x + y).$$

   Since x and y are positive integers, we must have $x - y = 1$ and $x + y = 1$. This means $2x = 2$, so $x = 1$. But then we must have $y = 0$ which is not a positive integer; this is a contradiction. Therefore the original statement is true. □

[104] courses/math1061/proof-techniques.pdf

§11 — Proof by Contraposition

Epp 4th ed. pp. 198–207 · 5th ed. (metric) pp. 218–228. Lecture 9 (week 3). Notes: proof-techniques[105].

Problems

1. Use proof by contraposition to prove the following statement:

   For all integers a and b, if $(ab)^2$ is odd then a is odd and b is odd.

2. Prove the following statement:

   For any integer n, n^2 is odd if and only if n is odd.

   Use a direct proof for the reverse implication ($\Leftarrow$), and a proof by contraposition for the forward implication ($\Rightarrow$).

3. Use proof by contraposition to prove the following statement:

   For all integers x and y, if $x^2(y^2 - 2y)$ is odd then x and y are odd.

Solutions

1. **For all integers a and b, if $(ab)^2$ is odd then a is odd and b is odd.**

   Proof. We prove the contrapositive statement, which is:

   > For all integers a and b, if a is even or b is even, then $(ab)^2$ is even.

   Let a and b be integers and suppose a is even or b is even. Without loss of generality, suppose a is even, so $a = 2k$ for some $k \in \mathbb{Z}$. Then $ab = 2kb$ is even, and therefore $(ab)^2 = 4k^2b^2 = 2(2k^2b^2)$. Since $k, b \in \mathbb{Z}$, we know $2k^2b^2 \in \mathbb{Z}$, so $(ab)^2$ is even by definition. □

2. **For any integer n, n^2 is odd if and only if n is odd.**

   Proof. To prove this "if and only if" statement, we must prove each of the following two statements:

   > ($\Leftarrow$) For any integer n, if n is odd, then n^2 is odd. ($\Rightarrow$) For any integer n, if n^2 is odd, then n is odd.

[105] courses/math1061/proof-techniques.pdf

($\Leftarrow$) Let n be an integer and suppose n is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. Hence $n^2 = (2k+1)^2 = 2(2k^2+2k)+1$. Since $k \in \mathbb{Z}$, we have $2k^2+2k \in \mathbb{Z}$ and hence n^2 is odd by definition.

($\Rightarrow$) The contrapositive is: for any integer n, if n is even, then n^2 is even. Let n be an integer and suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Hence $n^2 = 4k^2 = 2(2k^2)$. Since $k \in \mathbb{Z}$, we have $2k^2 \in \mathbb{Z}$, so n^2 is even by definition.

Therefore, for all integers n, n^2 is odd if and only if n is odd. □

3. **For all integers x and y, if $x^2(y^2-2y)$ is odd then x and y are odd.**

Proof. We will prove the contrapositive statement, which is:

> For all integers x and y, if x is even or y is even, then $x^2(y^2-2y)$ is even.

Let $x, y \in \mathbb{Z}$ and suppose x or y is even. We consider the cases of x even or y even individually.

Case 1. Suppose x is even. Then $x = 2k$ for some $k \in \mathbb{Z}$. Hence

$$x^2(y^2-2y) = (2k)^2(y^2-2y) = 2(2k^2)(y^2-2y).$$

Since $k, y \in \mathbb{Z}$, we have $(2k^2)(y^2-2y) \in \mathbb{Z}$, and hence $x^2(y^2-2y)$ is even, by definition.

Case 2. Suppose y is even. Then $y = 2\ell$ for some $\ell \in \mathbb{Z}$. Hence

$$x^2(y^2-2y) = x^2((2\ell)^2-2(2\ell)) = 2(x^2(2\ell^2-2\ell)).$$

Since $x, \ell \in \mathbb{Z}$, we have $x^2(2\ell^2-2\ell) \in \mathbb{Z}$ and thus $x^2(y^2-2y)$ is even, by definition. □

§12 — Rational Numbers

Epp 4th ed. pp. 163–170 · 5th ed. (metric) pp. 183–190. Lecture 9 (week 3). Notes: rational-and-irrational-numbers[106].

[106] courses/math1061/rational-and-irrational-numbers.pdf

Problems

1. Prove the following statement:

   For real numbers m and n, if m is irrational and n is rational, then $m-n$ is irrational.

2. Prove the following statement:

   For a real number r, if $2r^2 - 3r$ is irrational, then r is irrational.

3. Prove the following statement:

   For all real numbers r, s, n, if r is irrational, s is rational and n is a positive integer, then $n(r-s)$ is irrati

4. Recall that $\sqrt{2}$ is irrational. Consider the statement

   For every rational number r, if $r \neq 0$, then $r\sqrt{2}$ is an irrational number.

   (a) Write down the negation of the statement.

   (b) Use a proof by contradiction to prove the statement.

Solutions

1. **For real numbers m and n, if m is irrational and n is rational, then $m-n$ is irrational.**

   Proof. Let m and n be real numbers and suppose (for a contradiction) that m is an irrational number and n is rational, but $m-n$ is rational. Since n is rational, we have $n = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$. Since $m-n$ is rational, we have $m - n = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ where $d \neq 0$.

   Notice that $m = (m-n) + n$. Hence

   $$m = (m-n) + n = \frac{c}{d} + \frac{a}{b} = \frac{bc + ad}{bd}.$$

   Since $a, b, c, d \in \mathbb{Z}$ we have $bc + ad \in \mathbb{Z}$ and $bd \in \mathbb{Z}$. Further, since $b \neq 0$ and $d \neq 0$, we have $bd \neq 0$. Thus, by definition, m is rational. But this contradicts the assumption that m is irrational. $\square$

2. **For a real number r, if $2r^2 - 3r$ is irrational, then r is irrational.**

   Proof. We prove the contrapositive statement: for a real number r, if r is rational, then $2r^2 - 3r$ is rational.

   Let r be a real number and suppose $r \in \mathbb{Q}$. Then $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$. So

   $$2r^2 - 3r = 2\left(\frac{a}{b}\right)^2 - 3\left(\frac{a}{b}\right) = \frac{2a^2 - 3ab}{b^2}.$$

   Since $a, b \in \mathbb{Z}$, we have $2a^2 - 3ab \in \mathbb{Z}$ and $b^2 \in \mathbb{Z}$. Further, $b^2 \neq 0$ because $b \neq 0$. This means $2r^2 - 3r$ is rational, by definition. □

3. **For all real numbers r, s, and n, if r is irrational, s is rational and n is a positive integer, then $n(r - s)$ is irrational.**

   Proof. Suppose, for a contradiction, that the statement is false. Then there exist real numbers r, s and n such that r is an irrational number, s is rational, n is a positive integer, but $n(r - s)$ is rational. Since s is rational, $s = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$. Since $n(r - s)$ is rational, we have $n(r - s) = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ where $d \neq 0$.

   Now,

   $$\begin{aligned} r - s &= n(r - s)\left(\frac{1}{n}\right) \quad \text{noting that } n \neq 0 \\ r - s &= \frac{c}{d} \cdot \frac{1}{n} \\ r &= s + \frac{c}{d} \cdot \frac{1}{n} \\ &= \frac{a}{b} + \frac{c}{dn} \\ &= \frac{adn + bc}{bdn} \end{aligned}$$

   Since $a, b, c, d \in \mathbb{Z}$, we have $adn + bc \in \mathbb{Z}$ and $bdn \in \mathbb{Z}$. Further, since $b \neq 0$, $d \neq 0$ and $n \neq 0$, we have $bdn \neq 0$. This means r is rational, by definition, which is a contradiction. □

4. **For every rational number r, if $r \neq 0$, then $r\sqrt{2}$ is an irrational number.**

   (a) **Negation:** There exists $r \in \mathbb{Q}$ such that $r \neq 0$ and $r\sqrt{2} \in \mathbb{Q}$.

   (b)

   Proof. Suppose, for a contradiction, that the original statement is false. Then there exists $r \in \mathbb{Q}$ such that $r \neq 0$ and $r\sqrt{2} \in \mathbb{Q}$. Since r is rational, $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$, and since $r \neq 0$, we have $a \neq 0$. Since we have assumed $r\sqrt{2}$ is rational, $r\sqrt{2} = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ where $d \neq 0$.

Now

$$\sqrt{2} = \frac{r\sqrt{2}}{r} = \left(r\sqrt{2}\right)\left(\frac{1}{r}\right) = \frac{c}{d} \cdot \frac{b}{a} = \frac{bc}{ad}.$$

Since $a, b, c, d \in \mathbb{Z}$, we know $bc \in \mathbb{Z}$ and $ad \in \mathbb{Z}$. Further, since $a \neq 0$ and $d \neq 0$, we have $ad \neq 0$. Thus, by definition, $\sqrt{2}$ is rational. However, this is a contradiction, as $\sqrt{2}$ is irrational. Thus, the original statement is true. □

§13 — Divisibility

Epp 4th ed. pp. 170–179 · 5th ed. (metric) pp. 190–199. Lecture 10 (week 4). Notes: divisibility-and-factorisation[107].

Problems

1. Find the unique factorisation (in standard form) of the integer 27 720.
2. Prove or disprove the statement: for any integers m, n, and q, if $m \mid n$ and $n \mid q$ then $m^2 \mid nq$.
3. Prove the following statements.
   (a) If k is any even integer and m is any odd integer, then $(k+2)^2 - (m-3)^2$ is divisible by 4.
   (b) The difference between the cube of two consecutive integers leaves the remainder 1 when it is divided by 6.
4. Use proof by contradiction to prove this statement:

   For all integers c, d and e, if $c \mid d$ and $c \nmid e$, then $c \nmid (d + e)$.

Solutions

1. By repeated division, we have $27\,720 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$.
2. **For any integers m, n, and q, if $m \mid n$ and $n \mid q$ then $m^2 \mid nq$.** The statement is **true**, and we will prove it.

[107] courses/math1061/divisibility-and-factorisation.pdf

Proof. Let m, n, and q be non-zero integers. Assume $m \mid n$ and $n \mid q$. Then, by definition, there exist integers a, b such that

$$n = ma \quad \text{and} \quad q = nb.$$

Then observe

$$nq = (ma)(nb) = ma \cdot (mab) = m^2(a^2b).$$

Since $a^2b \in \mathbb{Z}$, we conclude that $m^2 \mid nq$. □

3. (a) **If k is any even integer and m is any odd integer, then $(k+2)^2 - (m-3)^2$ is divisible by 4.**

Proof. Let k be even and m be odd. Then $k = 2a$ and $m = 2b + 1$ for some $a, b \in \mathbb{Z}$. Hence

$$(k+2)^2 = (2a+2)^2 = 4(a+1)^2 \quad \text{and} \quad (m-3)^2 = (2b-2)^2 = 4(b-1)^2.$$

Therefore

$$(k+2)^2 - (m-3)^2 = 4\big[(a+1)^2 - (b-1)^2\big].$$

Since $a, b \in \mathbb{Z}$ we have $(a+1)^2 - (b-1)^2 \in \mathbb{Z}$ so $4 \mid \big[(k+2)^2 - (m-3)^2\big]$. □

(b) **The difference between the cube of two consecutive integers leaves the remainder 1 when it is divided by 6.**

Proof. Let n and $n+1$ be two consecutive integers. The difference between their cubes is

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1 = 3n(n+1) + 1.$$

Now whether n is odd or even (so whatever integer n is), one of n, $n+1$ will be odd and the other one will be even. Thus the product $n(n+1)$ is even.

Since $n(n+1)$ is even, write $n(n+1) = 2t$ for some $t \in \mathbb{Z}$. Then

$$(n+1)^3 - n^3 = 3(2t) + 1 = 6t + 1 \equiv 1 \pmod 6.$$

Hence the remainder is 1 upon division by 6. □

4. **For all integers c, d and e, if $c \mid d$ and $c \nmid e$, then $c \nmid (d+e)$.**

Proof. Suppose for a contradiction that the statement is false. Then there exist integers c, d and e such that $c \mid d$ and $c \nmid e$ and $c \mid (d+e)$.

Now $c \mid d$ means that $d = cx$ for some integer x. And $c \mid (d+e)$ means that $d + e = cy$ for some integer y.

Hence $d + e = cx + e = cy$, so $e = cy - cx = c(y - x)$, and we have $y - x \in \mathbb{Z}$, because x and y are both integers. Thus $c \mid e$, a contradiction. So the original statement is true. □

§14 — Modular Arithmetic

Epp 4th ed. pp. 482–485 · 5th ed. (metric) pp. 528–531. Lecture 11 (week 4). Notes: modular-arithmetic[108].

Problems

1. Prove the following statement:

   If y is any integer, then when y^2 is divided by 5, the remainder is always 0, 1 or 4, and never 2 or 3.

2. (a) Evaluate $58 \operatorname{div} 11$ and $58 \bmod 11$.

   (b) Find integers q and r, with $0 \leqslant r < d$ and $n = dq + r$, in the case that $n = -95$ and $d = 11$.

3. (a) An integer n, when divided by 7, leaves remainder 4. What is the remainder (between 0 and 6 inclusive) when $3n$ is divided by 7?

   (b) When the integer m is divided by 11, the remainder 9 is left. What remainder is left when $4m$ is divided by 22?

4. Prove or disprove the following statements.

   (a) For all real numbers x, $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.

   (b) For all odd integers n, $\left\lfloor \dfrac{n^2}{4} \right\rfloor = \left(\dfrac{n-1}{2}\right)\left(\dfrac{n+1}{2}\right)$.

Solutions

1. **If y is any integer, then when y^2 is divided by 5, the remainder is always 0, 1 or 4, and never 2 or 3.**

   Proof. Fix $y \in \mathbb{Z}$. By the division algorithm, $y = 5m + r$ for some $m \in \mathbb{Z}$ and $r \in \{0, 1, 2, 3, 4\}$. Then

   $$y^2 = (5m + r)^2 = 25m^2 + 10mr + r^2 \equiv r^2 \pmod 5.$$

   Let us compute the possible values for $r^2 \bmod 5$:

   $$0^2 \equiv 0, \quad 1^2 \equiv 1, \quad 2^2 \equiv 4, \quad 3^2 \equiv 9 \equiv 4, \quad 4^2 \equiv 16 \equiv 1 \pmod 5.$$

   Thus $y^2 \equiv 0, 1$, or $4 \pmod 5$, and never 2 or 3. □

[108] courses/math1061/modular-arithmetic.pdf

2. (a) Since $58 = 11 \cdot 5 + 3$ with $0 \le 3 < 11$, we have

$$58 \operatorname{div} 11 = 5 \quad \text{and} \quad 58 \bmod 11 = 3.$$

(b) We want $-95 = 11q + r$ with $0 \le r < 11$. Taking $q = -9$ gives

$$11(-9) = -99 \quad \Rightarrow \quad -95 = -99 + 4 = 11(-9) + 4,$$

so $q = -9$ and $r = 4$.

3. (a) If n leaves remainder 4 upon division by 7, then $n = 7k + 4$ for some integer k. Hence

$$3n = 21k + 12 = 7(3k + 1) + 5,$$

so the remainder when $3n$ is divided by 7 is **5**. (The source PDF prints this step as $7(3k) + 5$, which equals $21k + 5$, not $21k + 12$ — the quotient is $3k + 1$, as above. The answer of 5 is unaffected.)

(b) If m leaves remainder 9 upon division by 11, then $m = 11k + 9$ for some integer k. Then

$$4m = 44k + 36 = 22(2k + 1) + 14,$$

so the remainder when $4m$ is divided by 22 is **14**.

4. (a) **For all real numbers x, $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.** The statement is **false**. For a counterexample, take $x = \frac{3}{2}$. Then

$$\lfloor x^2 \rfloor = \left\lfloor \frac{9}{4} \right\rfloor = 2 \quad \text{but} \quad \lfloor x \rfloor^2 = \left\lfloor \frac{3}{2} \right\rfloor^2 = 1^2 = 1.$$

(b) **For all odd integers n, $\left\lfloor \dfrac{n^2}{4} \right\rfloor = \left(\dfrac{n-1}{2}\right)\left(\dfrac{n+1}{2}\right)$.** The statement is **true**.

Proof. Let n be an odd integer, so $n = 2k + 1$ for some integer k. Then

$$\frac{n^2}{4} = \frac{(2k+1)^2}{4} = \frac{4k^2 + 4k + 1}{4} = k(k+1) + \frac{1}{4},$$

so $\left\lfloor \frac{n^2}{4} \right\rfloor = k(k+1)$. Moreover, notice

$$\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) = \left(\frac{2k}{2}\right)\left(\frac{2k+2}{2}\right) = k(k+1).$$

Hence $\left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)$ for all odd integers n. □

§15 — The Euclidean Algorithm

Epp 4th ed. pp. 220–224 · 5th ed. (metric) pp. 250–254. Lecture 12 (week 4). Notes: gcd-lcm-and-euclidean-algorithm[109].

Problems

1. Use the Euclidean Algorithm to determine $\gcd(14, 3003)$.
2. Compute
   (a) $\operatorname{lcm}(12, 18)$
   (b) $\operatorname{lcm}(2^2 \cdot 3 \cdot 5,\ 2^3 \cdot 3^2)$
   (c) $\operatorname{lcm}(2800, 6125)$
3. Prove that for all positive integers a and b, we have
$$\gcd(a, b) = \operatorname{lcm}(a, b) \iff a = b.$$
4. Prove that for all positive integers a and b, we have
$$a \mid b \iff \operatorname{lcm}(a, b) = b.$$
5. (a) Use the Euclidean algorithm to determine $\gcd(931, 301)$. Show your working.
   (b) Write down the unique prime factorisation of 931 in standard form.
   (c) How many positive divisors does 931 have? (Hint: use part (b).)
6. (a) Using the Euclidean algorithm, compute $\gcd(116, 88)$. Show your working.
   (b) Using your answer to part (a), compute $\operatorname{lcm}(116, 88)$.

[109] courses/math1061/gcd-lcm-and-euclidean-algorithm.pdf

Solutions

1. We apply the Euclidean Algorithm:

$$3003 = 14 \cdot 214 + 7, \qquad 14 = 7 \cdot 2 + 0.$$

Hence $\gcd(14, 3003) = 7$.

2. (a) We factor $12 = 2^2 \cdot 3$ and $18 = 2 \cdot 3^2$ so $\operatorname{lcm}(12, 18) = 2^2 \cdot 3^2 = 36$.

(b) Using the maximum exponents of each prime, we have

$$\operatorname{lcm}(2^2 \cdot 3 \cdot 5,\ 2^3 \cdot 3^2) = 2^3 \cdot 3^2 \cdot 5 = 360.$$

(c) We factor $2800 = 2^4 \cdot 5^2 \cdot 7$ and $6125 = 5^3 \cdot 7^2$. Thus

$$\operatorname{lcm}(2800, 6125) = 2^4 \cdot 5^3 \cdot 7^2 = 98000.$$

3. **For all positive integers a and b, $\gcd(a, b) = \operatorname{lcm}(a, b)$ if and only if $a = b$.**

Proof. Let a and b be positive integers.

($\Longleftarrow$) It is clear that if $a = b$ then $\gcd(a, b) = \operatorname{lcm}(a, b)$.

($\Longrightarrow$) Assume $\gcd(a, b) = \operatorname{lcm}(a, b)$. Denoting $g = \gcd(a, b)$, we have $a = gm$ and $b = gn$ for some positive integers m and n with $\gcd(m, n) = 1$.

Now notice gm and gn have no common prime factors beyond those in g, so the least common multiple uses all primes from both m and n. Therefore

$$\operatorname{lcm}(a, b) = gmn.$$

Now, we have $g = gmn$. Therefore $mn = 1$, and so $m = n = 1$. This lets us conclude $a = b$. □

4. **For all positive integers a and b, $a \mid b$ if and only if $\operatorname{lcm}(a, b) = b$.**

Proof. Let us denote $g = \gcd(a, b)$ so that $a = gm$ and $b = gn$ for some positive integers m and n with $\gcd(m, n) = 1$. As explained in the previous question, we have

$$\operatorname{lcm}(a, b) = gmn.$$

($\Longrightarrow$) If $a \mid b$, then $gm \mid gn$, hence $m \mid n$. Since $\gcd(m, n) = 1$, we conclude $m = 1$. Therefore $\operatorname{lcm}(a, b) = gmn = gn = b$.

($\Longleftarrow$) If $\operatorname{lcm}(a, b) = b$, then $gmn = gn$ which means $m = 1$. Therefore $a = gm = g$ and so $a \mid b$. □

5. (a) Using the Euclidean Algorithm:

$$\begin{aligned} 931 &= 301 \cdot 3 + 28 \\ 301 &= 28 \cdot 10 + 21 \\ 28 &= 21 \cdot 1 + 7 \\ 21 &= 7 \cdot 3 + 0. \end{aligned}$$

Hence $\gcd(931, 301) = 7$.

(b) Since $\gcd(931, 301) = 7$, we know $7 \mid 931$, and we have $931 = 7 \cdot 133$. The number 133 has prime factorisation $7 \cdot 19$. Therefore the unique prime factorisation of 931 is $7^2 \cdot 19$.

(c) We have $931 = 7^2 \cdot 19^1$ from part (b). Any divisor of 931 must have a prime factorisation of the form $7^a \cdot 19^b$ where $a \in \{0, 1, 2\}$ and $b \in \{0, 1\}$. Thus the number of positive divisors is

$$(2 + 1)(1 + 1) = 6.$$

(There are 3 choices for the exponent a and 2 choices for the exponent b.)

6. (a) Using the Euclidean Algorithm:

$$\begin{aligned} 116 &= 88 \cdot 1 + 28 \\ 88 &= 28 \cdot 3 + 4 \\ 28 &= 4 \cdot 7 + 0 \end{aligned}$$

Hence $\gcd(116, 88) = 4$.

(b) Using $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$ for $a, b \in \mathbb{Z}^+$,

$$\operatorname{lcm}(116, 88) = \frac{116 \cdot 88}{\gcd(116, 88)} = \frac{116 \cdot 88}{4} = 116 \cdot 22 = 2552.$$

Application — Trace Tables for Algorithms

Not assessable for MATH1061. For reference, see S. Epp, Discrete Mathematics with Applications, *Section 4.8 (4th edition) or Section 4.10 (5th edition), Brooks/Cole, Belmont, CA.*

The following pseudo-code describes the **Division Algorithm**, with annotations to explain what each step is doing.

Input: a (a non-negative integer), d (a positive integer). This algorithm will take two inputs, a and d, and will find the quotient $a \operatorname{div} d$ and the remainder $a \bmod d$.

Algorithm body:

```
r := a          -- defining our variables
q := 0
                -- starting with a, repeatedly subtract lots of d until we reach
                -- a number less than d; each round adds 1 to the quotient count
while r >= d,
    r := r - d
    q := q + 1
end while
```

Output: q, r. Once we have obtained a number less than d, we are finished. The output q is the quotient ($a \operatorname{div} d$) and the output r is the remainder ($a \bmod d$).

The following pseudo-code describes the **Euclidean Algorithm**, with annotations to explain what each step is doing.

Input: A, B (integers with $A > B \geq 0$).

Algorithm body:

```
a := A          -- defining our variables
b := B
r := B
                -- if b /= 0, compute a mod b (call the algorithm above for this
                -- step) and set r to be this value; replace a with b and b with r
                -- and repeat until b = 0
while b /= 0,
    r := a mod b
    a := b
    b := r
end while
gcd := a
```

Output: gcd (a positive integer).

A **trace table** keeps track of the value of each variable at each step of an algorithm. For example, the trace table for the Division Algorithm with input variables $a = 19$ and $d = 4$ is:

Variable / iteration	0	1	2	3	4
a	19				
d	4				
r	19	15	11	7	3
q	0	1	2	3	4

Output: $q = 4$, $r = 3$.

Application problems

1. Construct a trace table to trace the action of the Division Algorithm for the following inputs.

   (i) $a = 26$, $d = 7$

   (ii) $a = 15$, $d = 6$

2. Construct a trace table to trace the action of the Euclidean Algorithm with inputs $A = 1001$ and $B = 871$.

Application solutions

1. (i) Trace table for the Division Algorithm with $a = 26$, $d = 7$:

Variable / iteration	0	1	2	3
a	26			
d	7			
r	26	19	12	5
q	0	1	2	3

   Output: $q = 3$, $r = 5$.

   (ii) Trace table for the Division Algorithm with $a = 15$, $d = 6$:

Variable / iteration	0	1	2
a	15		
d	6		
r	15	9	3
q	0	1	2

Output: $q = 2$, $r = 3$.

2. Trace table for the Euclidean Algorithm with inputs $A = 1001$ and $B = 871$:

Variable / iteration	0	1	2	3	4	5
A	1001					
B	871					
a	1001	871	130	91	39	13
b	871	130	91	39	13	0
r	871	130	91	39	13	0
gcd						13

Output: gcd = 13.

§16 — Sequences

Epp 4th ed. pp. 227–244 · 5th ed. (metric) pp. 258–275. Lecture 13 (week 5).

Problems

1. Write down the first four terms in the sequence defined by $a_k = \dfrac{k}{10+k}$ for $k = 1, 2, \ldots$.
2. Consider the sequences $a_k = 2k + 1$ and $b_k = (k-1)^3 + k + 2$ where $k = 0, 1, 2, \ldots$. Show that the first three terms of these sequences are identical, but that their fourth terms differ.
3. (a) Find an explicit formula for a sequence c_n whose first six terms are $-1, 1, -1, 1, -1, 1$.

   (b) Find an explicit formula for a sequence d_n whose first six terms are $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$.
4. (a) Write $(x^2-1)+(2x^3-1)+(3x^4-1)+(4x^5-1)+(5x^6-1)$ in summation notation.

   (b) Evaluate $\displaystyle\prod_{i=1}^{5}(-1)^i \cdot i$.

   (c) Write $(y-1)(y^2-2)(y^3-3)(y^4-4)$ in product notation.

(d) Evaluate $\displaystyle\sum_{j=1}^{n}(-1)^j\frac{1}{2^j}$ when $n=5$.

Solutions

1. We have

$$a_1=\frac{1}{11},\qquad a_2=\frac{2}{12}=\frac{1}{6},\qquad a_3=\frac{3}{13},\qquad a_4=\frac{4}{14}=\frac{2}{7}.$$

2. We compute

$$a_0=1,\quad a_1=3,\quad a_2=5,\quad a_3=7;$$

$$b_0=(-1)^3+0+2=1,\quad b_1=0^3+1+2=3,\quad b_2=1^3+2+2=5,\quad b_3=2^3+3+2=13.$$

Hence the first three terms agree ($a_0=b_0$, $a_1=b_1$ and $a_2=b_2$), but the fourth terms differ ($a_3=7$ while $b_3=13$).

3. (a) One explicit formula is

$$c_n=(-1)^n\qquad n=1,2,3,...$$

(b) The denominators are square numbers, so an explicit formula is

$$d_n=\frac{n}{(n+1)^2}\qquad n=1,2,3,...$$

4. (a) We have

$$(x^2-1)+(2x^3-1)+(3x^4-1)+(4x^5-1)+(5x^6-1)=\sum_{k=1}^{5}\left(kx^{k+1}-1\right).$$

(b) We have

$$\prod_{i=1}^{5}(-1)^i\cdot i=\left(\prod_{i=1}^{5}(-1)^i\right)\left(\prod_{i=1}^{5}i\right)=(-1)^{1+2+3+4+5}\cdot 5!=(-1)^{15}\cdot 120=-120.$$

(c) We have

$$(y-1)(y^2-2)(y^3-3)(y^4-4) = \prod_{k=1}^{4} (y^k - k)\,.$$

(d) When $n = 5$, we have

$$\sum_{j=1}^{5}(-1)^j \frac{1}{2^j} = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} = -\frac{11}{32}.$$

§17 — Mathematical Induction

Epp 4th ed. pp. 244–268 · 5th ed. (metric) pp. 275–301. Lecture 13 (week 5).

Problems

1. Use mathematical induction to prove the following statements.

   (a) For every positive integer n, $n(n+1)$ is even.

   (b) For every integer $n \geq 2$, $n^3 - n$ is a multiple of 6.

   (c) For all integers $n \geq 7$, $3^n < n!$.

   (d) For each integer $n \geq 1$,

   $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

   (e) For each integer $n \geq 0$, $1 + 4n \leq 5^n$.

Solutions

1. (a) **For every positive integer n, $n(n+1)$ is even.**

   Proof. Let $P(n)$ be the predicate "$n(n+1)$ is even".

   Basis: when $n = 1$, $n(n+1) = 2$ which is even, so $P(1)$ is true.

   (The inductive step is to show that for all integers $k \geq 1$, $P(k)$ implies $P(k+1)$.)

   Inductive hypothesis: suppose that for some integer $k \geq 1$, $P(k)$ is true. Thus $k(k+1) = 2a$ for some integer a.

Now consider $n = k + 1$. Then

$$(k+1)(k+2) = k(k+1) + 2(k+1) = 2a + 2(k+1) = 2(a+k+1).$$

Since a and k are integers, $a + k + 1$ is an integer, so $(k+1)(k+2)$ is even and $P(k+1)$ is true.

Thus, by the principle of mathematical induction, $n(n+1)$ is even for every positive integer n. □

(b) **For every integer $n \geq 2$, $n^3 - n$ is a multiple of 6.**

Proof. Let $P(n)$ be the predicate $6 \mid (n^3 - n)$.

Basis: when $n = 2$, $n^3 - n = 8 - 2 = 6$. Since $6 \mid 6$, $P(2)$ is true.

(The inductive step is to show that for all integers $k \geq 2$, $P(k)$ implies $P(k+1)$.)

Inductive hypothesis: suppose that for some integer $k \geq 2$, $P(k)$ is true. Thus $k^3 - k = 6a$ for some integer a.

Now consider $n = k + 1$. Then

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 - k + 3k^2 + 3k = 6a + 3k(k+1).$$

For the integer k, $k(k+1)$ is even from part (a), so let $k(k+1) = 2b$ for some integer b. Hence $(k+1)^3 - (k+1) = 6a + 3(2b) = 6(a+b)$. Thus $6 \mid ((k+1)^3 - (k+1))$ and so $P(k+1)$ is true.

Thus, by the principle of mathematical induction, $n^3 - n$ is a multiple of 6 for each integer $n \geq 2$. □

(c) **For all integers $n \geq 7$, $3^n < n!$.**

Proof. Let $P(n)$ be the predicate $3^n < n!$.

Basis: $P(7)$ is the statement $3^7 < 7!$. The left-hand side is $3^7 = 2187$ and the right-hand side is $7! = 5040$, so $P(7)$ holds.

(The inductive step is to show that for all integers $k \geq 7$, $P(k)$ implies $P(k+1)$.)

Inductive hypothesis: let $k \geq 7$ be an integer and assume that $P(k)$ is true; that is $3^k < k!$. We will now show that $P(k+1)$ holds, so we will show $3^{k+1} < (k+1)!$.

Taking the right-hand side, we have

$$\begin{aligned}
(k+1)! &= (k+1) \cdot k! \\
&> (k+1) \cdot 3^k && \text{by the inductive hypothesis} \\
&> 3 \cdot 3^k && \text{since } k \geq 7 \\
&= 3^{k+1}
\end{aligned}$$

Hence $P(k+1)$ is true.

Thus by the principle of mathematical induction, for all integers $n \geq 7$, $3^n < n!$. $\square$

(d) **For each integer $n \geq 1$, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.**

Proof. Let $P(n)$ be the predicate

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Basis: consider $n = 1$. We have $\sum_{i=1}^{1} i^3 = 1^3 = 1$ and $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$ so $P(1)$ is true.

(The inductive step is to show that for all integers $k \geq 1$, $P(k)$ implies $P(k+1)$.)

Inductive hypothesis: suppose $k \geq 1$ is an integer and $P(k)$ is true. That is,

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}.$$

Now consider $n = k + 1$.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 && \text{(splitting off the last term)} \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 && \text{by the inductive hypothesis} \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{(k+1)^2}{4}(k^2 + 4(k+1)) \\
&= \frac{(k+1)^2}{4}(k^2 + 4k + 4) \\
&= \frac{(k+1)^2}{4}(k+2)^2
\end{aligned}$$

Thus $P(k+1)$ is true.

Therefore, by the principle of mathematical induction, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for every integer $n \geq 1$. $\square$

(e) **For each integer $n \geq 0$, $1 + 4n \leq 5^n$.**

Proof. Let $P(n)$ be the predicate $1 + 4n \leq 5^n$.

Basis: when $n = 0$ we have $1 + 4 \cdot 0 \leq 5^0$, so $P(0)$ is true.

(The inductive step is to show that for all integers $k \geq 0$, $P(k)$ implies $P(k+1)$.)

Inductive hypothesis: suppose $k \geq 0$ is an integer and $P(k)$ is true; that is, $1 + 4k \leq 5^k$.

Now consider $n = k + 1$. Starting with the left-hand side of $P(k + 1)$, we have

$$\begin{aligned} 1 + 4(k+1) &= 1 + 4k + 4 \\ &\leq 5^k + 4 && \text{by the inductive hypothesis} \\ &\leq 5^k + 4 \cdot 5^k && \text{since } k \geq 0 \text{ so } 5^k \geq 1 \\ &= 5(5^k) \\ &= 5^{k+1} \end{aligned}$$

so $P(k + 1)$ is true.

Thus, by the principle of mathematical induction, $1 + 4n \leq 5^n$ for every integer $n \geq 0$. □

§18 — Strong Induction and the Well Ordering Principle

Epp 4th ed. pp. 268–279 · 5th ed. (metric) pp. 301–314. Lecture 14 (week 5).

Problems

1. Consider the statement

   For every integer $n \geq 2$, there exist integers $a, b \geq 0$ such that $n = 2a + 3b$.

   (a) Prove the statement using the well-ordering principle.

   (b) Prove the statement using strong mathematical induction.

2. The Lucas sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = 1$, $a_2 = 3$ and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geqslant 3$. Use strong mathematical induction to prove that $a_n \leqslant \left(\frac{7}{4}\right)^n$ for all $n \in \mathbb{Z}^+$.

3. Suppose x is a real number and $x + \frac{1}{x}$ is an integer. Use strong mathematical induction to prove $a_n := x^n + \frac{1}{x^n}$ is an integer for all $n \geq 0$.

Solutions

1. **For every integer $n \geq 2$, there exist integers $a, b \geq 0$ such that $n = 2a + 3b$.**

   (a)

   Proof (using the well-ordering principle). Consider the set of counterexamples

   $$S = \{\, n \in \mathbb{Z}^{\geq 2} : \text{there do not exist integers } a, b \geq 0 \text{ with } n = 2a + 3b \,\}.$$

   If S is empty, we are done. Otherwise, by the well-ordering principle, S has a least element. Let us call it N.

   Notice that $2 \notin S$ because $2 = 2 \cdot 1 + 3 \cdot 0$, and $3 \notin S$ because $3 = 2 \cdot 0 + 3 \cdot 1$. Also $4 \notin S$ because $4 = 2 \cdot 2 + 3 \cdot 0$. Hence $N \geq 5$. By minimality of N, we must have $N - 2 \notin S$. Therefore there exist integers $a, b \geq 0$ such that

   $$N - 2 = 2a + 3b.$$

   In other words, we have

   $$N = 2(a + 1) + 3b.$$

   This means $N \notin S$, contradicting the assumption that $N \in S$. Hence S is empty, and the statement holds for all $n \geq 2$. $\square$

   (b)

   Proof (using strong induction). Let $P(n)$ be the predicate: "there exist integers $a, b \geq 0$ such that $n = 2a + 3b$."

   Basis: we have

   $$2 = 2 \cdot 1 + 3 \cdot 0, \qquad 3 = 2 \cdot 0 + 3 \cdot 1, \qquad 4 = 2 \cdot 2 + 3 \cdot 0,$$

   so $P(2)$, $P(3)$, $P(4)$ are true.

   (The inductive step is to show that for every integer $k \geq 4$, if $P(2), P(3), \ldots, P(k)$ are all true then $P(k + 1)$ is true.)

   Inductive hypothesis: suppose $k \geq 4$ is an integer and assume $P(m)$ is true for every integer $m = 2, 3, \ldots, k$.

   Consider $n = k + 1$. Since $k \geq 4$, we have $k - 1 \geq 3$, so by the inductive hypothesis $P(k - 1)$ is true. Thus there exist $a, b \geq 0$ such that

   $$k - 1 = 2a + 3b.$$

   Adding 2 gives

   $$k + 1 = 2(a + 1) + 3b,$$

   so $P(k + 1)$ holds. $\square$

2. **The Lucas sequence** $a_1, a_2, a_3, ...$ is defined by $a_1 = 1$, $a_2 = 3$ and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geqslant 3$. Prove that $a_n \leqslant \left(\frac{7}{4}\right)^n$ for all $n \in \mathbb{Z}^+$.

   Proof. Let $P(n)$ be the predicate $a_n \leqslant \left(\frac{7}{4}\right)^n$.

   Basis: when $n = 1$, $a_1 = 1 \leqslant \left(\frac{7}{4}\right)^1 = 1\frac{3}{4}$ is true, so $P(1)$ is true. When $n = 2$, $P(2)$ holds because $a_2 = 3 \leqslant \left(\frac{7}{4}\right)^2 = \frac{49}{16} = 3\frac{1}{16}$.

   (The inductive step is to show that for every integer $k \geq 1$, if $P(1), ..., P(k)$ are all true then $P(k+1)$ is true.)

   Inductive hypothesis: suppose $k \geq 1$ is an integer and that $P(m)$ is true for all m with $1 \leq m \leq k$.

   Now consider $P(k+1)$, which we want to prove holds. We have

$$\begin{aligned}
a_{k+1} &= a_k + a_{k-1} \\
&\leqslant \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} && \text{by the inductive hypothesis} \\
&= \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4} + 1\right) \\
&= \left(\frac{7}{4}\right)^{k-1}\left(\frac{11}{4}\right) \\
&\leqslant \left(\frac{7}{4}\right)^{k-1}\left(\frac{49}{16}\right) && \text{noting } \tfrac{11}{4} = \tfrac{44}{16} < \tfrac{49}{16} \\
&= \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^2 \\
&= \left(\frac{7}{4}\right)^{k+1}.
\end{aligned}$$

   Hence $P(k+1)$ holds. Therefore, by the principle of strong mathematical induction, $a_n \leqslant \left(\frac{7}{4}\right)^n$ for all integers $n \geqslant 1$. □

3. **Suppose x is a real number and $x + \frac{1}{x}$ is an integer. Prove $a_n := x^n + \frac{1}{x^n}$ is an integer for all $n \geq 0$.**

   Proof. Suppose x is a real number and $x + \frac{1}{x}$ is an integer. Let $P(n)$ be the predicate "a_n is an integer."

   Basis: $P(0)$ holds since $a_0 = x^0 + \frac{1}{x^0} = 2$. $P(1)$ holds since $a_1 = x + \frac{1}{x}$, which is an integer by assumption.

   (The inductive step is to show that for every integer $k \geq 1$, if $P(0), P(1), ..., P(k)$ are all true then $P(k+1)$ is true.)

Inductive hypothesis: suppose $k \geq 1$ is an integer and $P(m)$ holds for all integers $m = 0, 1, \ldots, k$.

We now aim to show that $P(k+1)$ is true. Notice that

$$\begin{aligned} a_1 a_k &= \left(x + \frac{1}{x}\right)\left(x^k + \frac{1}{x^k}\right) \\ &= \left(x^{k+1} + \frac{1}{x^{k+1}}\right) + \left(x^{k-1} + \frac{1}{x^{k-1}}\right) = a_{k+1} + a_{k-1}. \end{aligned}$$

Therefore $a_{k+1} = a_1 a_k - a_{k-1}$. By the inductive hypothesis, a_k and a_{k-1} are integers, and a_1 is an integer, so a_{k+1} is an integer. Hence $P(k+1)$ holds.

Therefore, by the principle of strong mathematical induction, a_n is an integer for all $n \geq 0$. □

§19 — Recursive Definitions

Epp 4th ed. pp. 290–304 · 5th ed. (metric) pp. 325–340. Lecture 15 (week 6).

Problems

1. Define the sequence $b_1 = 2$, $b_2 = 6$ and

$$b_k = b_{k-1} + b_{k-2} + \gcd(b_{k-1}, b_{k-2})$$

for all $k \geq 3$. Use strong induction to prove b_n is even for all $n \geq 1$.

2. A sequence is defined by $a_1 = 2$ and

$$a_k = k(a_{k-1} + k - 1)$$

for all integers $k \geq 2$. Use mathematical induction to prove a_n is even for all $n \geq 1$.

3. Recursively define the set S of binary strings as follows.

   (a) *(Base rule)* $1 \in S$.

   (b) *(Recursion rule)* If $s \in S$ then $0s \in S$ and $1s \in S$ (e.g. if $s = 1$ then $0s = 01$).

   (c) *(Restriction rule)* No other strings lie in S besides those given by the rules above.

   What are the strings contained in S arising from the first three applications of the recursive rule?

Solutions

1. *Proof.* Let $P(n)$ be the predicate "b_n is even".

   Base case. The base cases are $n = 1, 2$. We have $b_1 = 2$ and $b_2 = 6$ so $P(1)$ and $P(2)$ are true.

   Inductive hypothesis. Assume $P(m)$ is true for all integers m with $1 \leq m \leq k$, where $k \geq 2$ is an integer. That is, we assume $b_1, b_2, \ldots, b_k$ are all even.

   Inductive step. We need to show that $P(k+1)$ is true, i.e. b_{k+1} is even. Using the recursive definition, we have

   $$b_{k+1} = b_k + b_{k-1} + \gcd(b_k, b_{k-1}).$$

   By the inductive hypothesis, b_k and b_{k-1} are even, so $2 \mid b_k$ and $2 \mid b_{k-1}$, hence $2 \mid \gcd(b_k, b_{k-1})$ as well. Therefore each term on the right-hand side is even, so their sum b_{k+1} is even. Thus $P(k+1)$ holds.

   Therefore, b_n is even for all integers $n \geq 1$. □

2. *Proof.* Let $P(n)$ be the predicate "a_n is even".

   Base case. The base case is $n = 1$. We have $a_1 = 2$, which is even, so $P(1)$ is true.

   Inductive hypothesis. Assume $P(k)$ is true for some integer $k \geq 1$. That is, assume a_k is even.

   Inductive step. We need to show that $P(k+1)$ is true, i.e. a_{k+1} is even. By the recursive definition, we have

   $$a_{k+1} = (k+1)(a_k + (k+1) - 1) = (k+1)(a_k + k).$$

   Note that a_k is even by the inductive hypothesis. We split into cases:

   - If k is even, then $a_k + k$ is even, hence a_{k+1} is even.
   - If k is odd, then $k+1$ is even, hence a_{k+1} is even.

   In all cases, a_{k+1} is even, so $P(k+1)$ is true.

   Therefore, a_n is even for all integers $n \geq 1$. □

3. By the base rule, we only know $1 \in S$.

   First application of the recursive rule:

   $$01 \in S, \qquad 11 \in S.$$

   Second application of the recursive rule:

$$001 \in S, \quad 101 \in S, \quad 011 \in S, \quad 111 \in S.$$

Third application of the recursive rule:

$$0001 \in S, \quad 1001 \in S, \quad 0101 \in S, \quad 1101 \in S,$$
$$0011 \in S, \quad 1011 \in S, \quad 0111 \in S, \quad 1111 \in S.$$

§20 — Solving Recurrence Relations

Epp 4th ed. pp. 304–328 · 5th ed. (metric) pp. 340–364. Lecture 16 (week 6).

Problems

1. Define a sequence $a_n = 2^n + 5 \cdot 3^n$ for all integers $n \geq 0$. Show that

$$a_n = 5a_{n-1} - 6a_{n-2}$$

for all integers $n \geq 2$.

2. Let $\{b_n\}_{n\geq 0}$ be the sequence defined by

$$b_0 = 2, \qquad b_1 = 5, \qquad \text{and} \qquad b_k = 2b_{k-1} - b_{k-2} \text{ for each integer } k \geq 2.$$

   (a) Write down the values of $b_0, b_1, b_2, b_3, b_4, b_5$.

   (b) Based on your answers for part (a), guess an explicit formula for this sequence.

   (c) Prove that your guess from part (b) is correct.

3. Let T_n be a sequence defined by $T_1 = 1$ and

$$T_{n+1} = 1 + (T_1 \times T_2 \times T_3 \times \cdots \times T_n)$$

for all $n \geq 1$. Prove that $T_{n+1} = (T_n)^2 - T_n + 1$ for all $n \geq 2$.

4. Consider the Fibonacci sequence defined recursively by

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_n &= a_{n-2} + a_{n-1} \qquad \text{for all integers } n \geq 2. \end{aligned}$$

Use strong mathematical induction to prove the explicit formula

$$a_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

for all integers $n \geq 0$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. (You may use the fact that $\phi^2 = \phi + 1$ and $\psi^2 = \psi + 1$.)

Solutions

1. **Define a sequence $a_n = 2^n + 5 \cdot 3^n$ for all integers $n \geq 0$. Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$.**

Proof. For $n \geq 2$, we have $n - 2 \geq 0$ so all three terms a_n, a_{n-1} and a_{n-2} are defined in the given formula. Then,

$$\begin{aligned} 5a_{n-1} - 6a_{n-2} &= 5\left(2^{n-1} + 5 \cdot 3^{n-1}\right) - 6\left(2^{n-2} + 5 \cdot 3^{n-2}\right) \\ &= \left(5 \cdot 2^{n-1} - 6 \cdot 2^{n-2}\right) + \left(25 \cdot 3^{n-1} - 30 \cdot 3^{n-2}\right) \\ &= (10 - 6)2^{n-2} + (75 - 30)3^{n-2} \\ &= 4 \cdot 2^{n-2} + 45 \cdot 3^{n-2} \\ &= 2^n + 5 \cdot 3^n \\ &= a_n. \end{aligned}$$

Hence $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$. □

2. (a)

$$\begin{aligned} b_0 &= 2 \\ b_1 &= 5 \\ b_2 &= 2(5) - 2 = 8 \\ b_3 &= 2(8) - 5 = 11 \\ b_4 &= 2(11) - 8 = 14 \\ b_5 &= 2(14) - 11 = 17 \end{aligned}$$

(b) Guess that $b_n = 3n + 2$ for each integer $n \geq 0$.

(c)

Proof. The guess gives $b_0 = 3(0) + 2 = 2$ and $b_1 = 3(1) + 2 = 5$ so it gives the correct initial conditions. Now we substitute the guess into the recurrence relation. The LHS of the recurrence relation is $b_k = 3k + 2$. The RHS of the recurrence relation is

$$\begin{aligned} 2b_{k-1} - b_{k-2} &= 2(3(k-1)+2) - (3(k-2)+2) \\ &= 2(3k-1) - (3k-4) \\ &= 6k - 2 - 3k + 4 \\ &= 3k + 2 \end{aligned}$$

So the guess also satisfies the recurrence relation. Thus the guess is correct. □

Alternative proof (by strong induction). Let $P(n)$ be the predicate $b_n = 3n + 2$.

Basis: $b_0 = 2$ from the recursive definition and $3(0) + 2 = 2$ so $P(0)$ is true. $b_1 = 5$ from the recursive definition and $3(1) + 2 = 5$ so $P(1)$ is true.

Inductive hypothesis: suppose $k \geq 1$ is an integer and $P(0), P(1), \ldots, P(k)$ are true.

Now

$$\begin{aligned} b_{k+1} &= 2b_k - b_{k-1} && \text{by the rec. rel. since } k+1 \geq 2 \\ &= 2(3k+2) - (3(k-1)+2) && \text{since } P(k) \text{ and } P(k-1) \text{ are true} \\ &= 6k + 4 - (3k - 1) \\ &= 3k + 5 \\ &= 3(k+1) + 2 \end{aligned}$$

So $P(k+1)$ is true. Therefore, by the principle of (strong) mathematical induction, the guess is correct for this recursively defined sequence. □

3. **Let T_n be a sequence defined by $T_1 = 1$ and $T_{n+1} = 1 + (T_1 \times \cdots \times T_n)$ for all $n \geq 1$. Prove that $T_{n+1} = (T_n)^2 - T_n + 1$ for all $n \geq 2$.**

Proof. We can prove this simply by applying the recursive formula twice, once for T_{n+1} and once for T_n:

$$\begin{aligned} T_{n+1} &= 1 + (T_1 \cdot T_2 \cdot T_3 \cdots T_n) && \text{by the recursive definition} \\ &= 1 + T_n \cdot (T_1 \cdot T_2 \cdot T_3 \cdots T_{n-1}) \\ &= 1 + T_n \cdot (T_n - 1) && \text{since } T_n = 1 + (T_1 \cdots T_{n-1}) \text{ by rec. def.} \\ &= 1 + (T_n)^2 - T_n \\ &= (T_n)^2 - T_n + 1. \end{aligned}$$

Note that both applications of the recursive definition are valid, since $n-1 \geq 1$. $\square$

4. **Fibonacci explicit formula.**

Proof. Let $P(n)$ be the predicate

$$a_n = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$

Basis. The base cases are $n = 0, 1$, which are true because

$$\frac{\phi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 = a_0,$$

and

$$\frac{\phi - \psi}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = a_1.$$

Inductive hypothesis. Now suppose that for some integer $m \geq 1$, we have that $P(i)$ is true for all $0 \leq i \leq m$. In particular, $P(m)$ and $P(m-1)$ are true, that is,

$$a_m = \frac{\phi^m - \psi^m}{\sqrt{5}} \qquad \text{and} \qquad a_{m-1} = \frac{\phi^{m-1} - \psi^{m-1}}{\sqrt{5}}.$$

We need to show that $P(m+1)$ is true. That is, we aim to show

$$a_{m+1} = \frac{\phi^{m+1} - \psi^{m+1}}{\sqrt{5}}.$$

Starting with the right-hand side of $P(m+1)$, we have

$$\begin{aligned}
\frac{\phi^{m+1} - \psi^{m+1}}{\sqrt{5}} &= \frac{\phi^2\phi^{m-1} - \psi^2\psi^{m-1}}{\sqrt{5}} \\
&= \frac{(\phi+1)\phi^{m-1} - (\psi+1)\psi^{m-1}}{\sqrt{5}} && \text{since } \phi^2 = \phi + 1 \text{ and } \psi^2 = \psi + 1 \\
&= \frac{(\phi^m + \phi^{m-1}) - (\psi^m + \psi^{m-1})}{\sqrt{5}} \\
&= \frac{\phi^m - \psi^m}{\sqrt{5}} + \frac{\phi^{m-1} - \psi^{m-1}}{\sqrt{5}} \\
&= a_m + a_{m-1} && \text{by the inductive hypothesis} \\
&= a_{m+1} && \text{by definition.}
\end{aligned}$$

Hence $P(m+1)$ is true. $\square$

§21 — Set Theory Definitions

Epp 4th ed. pp. 336–351 · 5th ed. (metric) pp. 377–391. Lecture 17 (week 6).

Problems

1. Define the sets

$$\begin{aligned} V &= \{x \in \mathbb{Z} \mid 2 \leqslant x \leqslant 11 \text{ and } x \text{ is odd}\}, \\ X &= \{x \in \mathbb{Z} \mid -1 \leqslant x \leqslant 4\}, \\ Y &= \{2, 3, 7, 9\}, \\ Z &= \{x \in \mathbb{Z} \mid x \neq 0\}. \end{aligned}$$

   Write down the following sets, listing their elements and using brackets as appropriate.

   (a) $V \cap X$. (b) $V \cup Y$. (c) $X \cap Z$.

   (b) $X \cap (Z - Y)$. (e) $(X \cup Y) \cap V$. (f) $X \cup (Y \cap V)$.

2. For each of the following sets, write down its elements and then state the size of the set; that is, find $|A|$, $|B|$, $|C|$ and $|D|$.

   (a) $A = \{x \in \mathbb{Z} \mid \exists\, i \in \mathbb{Z} \text{ such that } x = 1 + (-1)^i\}$.

   (b) $B = \{y \in \mathbb{Z} \mid -8 < y \leqslant -5\}$.

   (c) $C = \{s \in \mathbb{Z} \mid s^2 + 1 = 0\}$.

   (d) $D = \{t \in \mathbb{Z} \mid t^4 = 1\}$.

3. Define the sets $A = \{0, 1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{1, 2, 4, 5, 7\}$. List the elements of the set

$$R = ((A - B) \cap (A - C)) \cup (B \cap C \cap A).$$

4. Are the following statements true or false? Justify in one or two sentences.

   (a) $x \in \{x\}$ (b) $\{x\} \subseteq \{x\}$ (c) $\{\emptyset\} \in \{\{\emptyset\}\}$

   (b) $\emptyset \subseteq \{x\}$ (e) $\emptyset \in \{x\}$

Solutions

1. (a) $V \cap X = \{3\}$

   (b) $V \cup Y = \{2, 3, 5, 7, 9, 11\}$

   (c) $X \cap Z = \{-1, 1, 2, 3, 4\}$

   (d) $X \cap (Z - Y) = \{-1, 1, 4\}$

   (e) $(X \cup Y) \cap V = \{3, 7, 9\}$

   (f) $X \cup (Y \cap V) = \{-1, 0, 1, 2, 3, 4, 7, 9\}$

2. (a) Note that $(-1)^i$ is either $+1$ or -1 (according as i is even or odd). So x is either $1 + 1$ or $1 - 1$, that is, 2 or 0. So $A = \{2, 0\}$ and $|A| = 2$.

   (b) $B = \{-7, -6, -5\}$, so $|B| = 3$.

   (c) If $s^2 + 1 = 0$, we have $s^2 = -1$, and there are no such real numbers s, let alone integers! So $C = \emptyset$ and $|C| = 0$.

   (d) $D = \{-1, 1\}$, since $t^4 = 1$ with $t \in \mathbb{Z}$ implies $t = \pm 1$. Hence $|D| = 2$.

3. We have $A - B = \{0, 1, 2\}$ and $A - C = \{0, 3\}$, so $(A - B) \cap (A - C) = \{0\}$. Also $B \cap C \cap A = \{4\}$. Thus $R = \{0\} \cup \{4\} = \{0, 4\}$.

4. (a) **True:** $\{x\}$ is the set whose only element is x, so $x \in \{x\}$.

   (b) **True:** every set is a subset of itself, so $\{x\} \subseteq \{x\}$.

   (c) **True:** $\{\{\emptyset\}\}$ has the single element $\{\emptyset\}$, so $\{\emptyset\} \in \{\{\emptyset\}\}$.

   (d) **True:** $\emptyset \subseteq S$ for every set S, in particular $\emptyset \subseteq \{x\}$.

   (e) **False** (in general): $\emptyset \in \{x\}$ would require $\emptyset = x$.

Application — Computer Representations of Sets

Not assessable for MATH1061. Taken from K. Rosen, Discrete Mathematics and its Applications, *7th edition, 1991, pages 134–138.*

There are various ways to represent sets using a computer. One such method involves the use of binary strings.

If we fix a finite universal set U with an arbitrary ordering, for example $U = \{a_1, a_2, \ldots, a_n\}$, we can represent subsets of U using binary strings of length n. A 1 in the ith position of the string indicates that element a_i is in the subset and a 0 indicates it is not.

For example if $U = \{1, 2, 3\}$ then the string 000 represents the empty set, 101 represents the subset $\{1, 3\}$ and 111 represents the entire set U.

The set operations union and intersection can be computed at the level of binary strings by utilising the bitwise OR and AND operations. The union of two sets corresponds to the bitwise OR operation on the strings and the intersection corresponds to bitwise AND.

Application problems

1. Define the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

   (a) Express each of the following subsets of U using binary strings:

   i. $\{3, 4, 5\}$

   ii. $\{1, 3, 6, 10\}$

   iii. $\{a \in U \mid a \text{ is prime}\}$.

   (b) Using the same U, find the set represented by the following strings:

   i. 1111001111

   ii. 0101111000

   iii. 1000000001

   (c) Use binary strings to compute the following set operations.

   i. $\{1, 3, 4, 6, 10\} \cap \{2, 3, 6, 9, 10\}$

   ii. $\{2, 4, 6, 8\} \cup \{3, 4, 5\}$

Application solutions

1. (a) i. 0011100000

   ii. 1010010001

   iii. 0110101000 — the primes in U are $2, 3, 5, 7$. (The source PDF prints 1010111000 here, which decodes to $\{1, 3, 5, 6, 7\}$ and is not the set of primes; the string above is the correct one.)

   (b) i. $\{1, 2, 3, 4, 7, 8, 9, 10\}$

   ii. $\{2, 4, 5, 6, 7\}$

   iii. $\{1, 10\}$

   (c) i. 1011010001 AND 0110010011 = 0010010001, so the intersection is $\{3, 6, 10\}$.

   ii. 0101010100 OR 0011100000 = 0111110100, so the union is $\{2, 3, 4, 5, 6, 8\}$.

§22 — More Definitions and Examples of Sets

Epp 4th ed. pp. 336–351 · 5th ed. (metric) pp. 377–391. Lecture 17 (week 6).

Problems

1. Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{3, 4\}$. List the elements of the following sets.
   (a) $A \cap C$
   (b) $(A \cup C) - B$
   (c) $(A \times B) \cap (A \times C)$
   (d) $\mathcal{P}(B \cap C)$
   (e) $\{x : (x \subseteq A) \wedge (x \in \mathcal{P}(B))\}$
2. Are the following statements true or false?
   (a) $\mathbb{Q} \in \mathbb{R}$.
   (b) $\{1, 2\} \subseteq \mathbb{Z}$.
   (c) $\{-3, 0\} \subseteq \mathcal{P}(\mathbb{Z})$.
   (d) $\{(1, 2), (0, 1)\} \in \mathbb{Z} \times \mathbb{Z}$.
3. Let $X = \mathcal{P}(\emptyset)$ and $Y = \mathcal{P}(\mathcal{P}(\emptyset))$. List the elements of each of the following sets.
   (a) X (b) Y (c) $X \cap Y$ (d) $X \cup Y$
4. Define sets $S = \{3, 5\}$ and $T = \{2, 3, 7\}$. Are the following statements true or false?
   (a) $S \subseteq T$
   (b) $\emptyset \in \mathcal{P}(S)$
   (c) $\emptyset \in T$
   (d) $(3, 3, 3) \in S \times T$

Solutions

1. (a) $A \cap C = \emptyset$.
   (b) $(A \cup C) - B = \{1, 4\}$.
   (c) $(A \times B) \cap (A \times C) = \{(1, 3), (2, 3)\}$.
   (d) $\mathcal{P}(B \cap C) = \{\emptyset, \{3\}\}$.
   (e) $\{x : (x \subseteq A) \wedge (x \in \mathcal{P}(B))\} = \{\emptyset, \{2\}\}$.
2. (a) **False.** ($\mathbb{Q}$ is a *subset* of $\mathbb{R}$, not an element of it.)
   (b) **True.**
   (c) **False.** (The elements of $\mathcal{P}(\mathbb{Z})$ are sets of integers; -3 and 0 are integers, not sets.)
   (d) **False.** (It is a *subset* of $\mathbb{Z} \times \mathbb{Z}$, not an element — the elements of $\mathbb{Z} \times \mathbb{Z}$ are ordered pairs.)
3. (a) $X = \mathcal{P}(\emptyset) = \{\emptyset\}$.
   (b) $Y = \mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$.
   (c) $X \cap Y = \{\emptyset\}$.
   (d) $X \cup Y = \{\emptyset, \{\emptyset\}\}$.
4. (a) **False.** ($5 \in S$ but $5 \notin T$.)
   (b) **True.** (The empty set is an element of every power set.)
   (c) **False.** (T's elements are 2, 3 and 7.)
   (d) **False.** (The elements of $S \times T$ are ordered *pairs*, not triples.)

§23 — Properties of Sets

Epp 4th ed. pp. 352–366 · 5th ed. (metric) pp. 391–407. Lecture 18 (week 7).

Problems

1. (a) Use the element method to prove that for all sets A and B

$$A \cup (B - A) = A \cup B.$$

   (b) Use the element method to prove that for all sets X, Y and Z,

$$(X - Z) \cap (Y - Z) \cap (X - Y) = \emptyset.$$

   (c) Use set identities to prove that for all sets A, B and C

$$(A - B) - C = A - (B \cup C).$$

2. Prove or disprove the following:

   (a) For all sets A and B, we have $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

   (b) For all sets A and B, we have $\mathcal{P}(A) = \mathcal{P}(B) \iff A = B$.

   (c) There exists a set A such that $\mathcal{P}(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.

   (d) Let A, B and C be sets. If $B \subseteq C$ then $A \times B \subseteq A \times C$.

3. Let A, B and C be sets. Prove that

$$A - (B \cup C) = (A - B) - C.$$

Solutions

1. (a) **For all sets A and B, $A \cup (B - A) = A \cup B$.**

   Proof. First we show that $A \cup (B - A) \subseteq A \cup B$.

   Consider an arbitrary element $x \in A \cup (B - A)$. Then we know $x \in A$ or $x \in B - A$. This means $x \in A$, or $x \in B$ and $x \notin A$. If $x \in A$, then $x \in A \cup B$. Similarly if $x \in B - A$, then it must be in B and so $x \in A \cup B$. In either case, we have $x \in A \cup B$, thus $A \cup (B - A) \subseteq A \cup B$.

   Now we show that $A \cup B \subseteq A \cup (B - A)$.

   Let $x \in A \cup B$, then $x \in A$ or $x \in B$ or it is in both A and B. If $x \in A$, then $x \in A \cup (B - A)$. If $x \notin A$ then $x \in B$ since $x \in A \cup B$. So we have that $x \in B$ and $x \notin A$, hence $x \in B - A$ and thus $x \in A \cup (B - A)$. This proves that $A \cup B \subseteq A \cup (B - A)$, and together this proves that $A \cup B = A \cup (B - A)$. $\square$

(b) **For all sets X, Y and Z, $(X - Z) \cap (Y - Z) \cap (X - Y) = \emptyset$.**

Proof. Assume that $(X - Z) \cap (Y - Z) \cap (X - Y) \neq \emptyset$.

So let $x \in (X - Z) \cap (Y - Z) \cap (X - Y)$. Then, in particular, $x \in Y - Z$ and $x \in X - Y$. The former implies $x \in Y$, but the latter implies $x \notin Y$. No such x exists. Therefore the set is empty. □

(c) **For all sets A, B and C, $(A - B) - C = A - (B \cup C)$.**

Proof. Using $A - B = A \cap B^c$, $(S - T) = S \cap T^c$, and De Morgan's law, we have

$$(A-B)-C = (A\cap B^c)\cap C^c = A\cap(B^c\cap C^c) = A\cap(B\cup C)^c = A-(B\cup C). \qquad \square$$

2. (a) **For all sets A and B, $\mathcal{P}(A\cup B) = \mathcal{P}(A)\cup\mathcal{P}(B)$. False.** Counterexample: $A = \{1\}$, $B = \{2\}$. Then

$$\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \neq \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \{\emptyset, \{1\}, \{2\}\}.$$

(b) **For all sets A and B, $\mathcal{P}(A) = \mathcal{P}(B) \iff A = B$. True.**

($\Longleftarrow$) If $A = B$ then clearly $\mathcal{P}(A) = \mathcal{P}(B)$.

($\Longrightarrow$) We next show that if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$. Suppose (for a contradiction) that $\mathcal{P}(A) = \mathcal{P}(B)$ but $A \neq B$. Then (without loss of generality), there exists an element x in A that is not in B. Therefore, there is an element $\{x\}$ in $\mathcal{P}(A)$ that is not in $\mathcal{P}(B)$. Therefore $\mathcal{P}(A) \neq \mathcal{P}(B)$ — a contradiction. Thus, for all sets A and B, if $\mathcal{P}(A) = \mathcal{P}(B)$ then $A = B$. □

(c) **There exists a set A such that $\mathcal{P}(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$. False.**

No such set exists. The power set of any set must contain the empty set — that is, $\emptyset \in \mathcal{P}(A)$, where A is any set. Also, the power set of any set of size n has cardinality 2^n. The power set above contains 7 elements, and there does not exist any integer n such that $2^n = 7$.

(d) **Let A, B and C be sets. If $B \subseteq C$ then $A \times B \subseteq A \times C$. True.**

First note that if $A = \emptyset$ then $A \times B = A \times C = \emptyset$ and the statement is true. Also, if $B = \emptyset$ then $A \times B = \emptyset$ and $\emptyset \subseteq A \times C$ for any C, so the statement is true.

Suppose A, B and C are non-empty sets with $B \subseteq C$. Thus if $b \in B$, then $b \in C$. Consider any element $(a, b) \in A \times B$, where $a \in A$ and $b \in B$. Since

$b \in B$, $b \in C$. As such, $a \in A$ and $b \in C$ for all a or b, so $(a, b) \in A \times C$. Therefore $A \times B \subseteq A \times C$. □

3. **For all sets A, B and C, $A - (B \cup C) = (A - B) - C$.**

   Proof Option 1. For any x, we have

   $$x \in A - (B \cup C) \iff x \in A \wedge x \notin (B \cup C) \iff x \in A \wedge (x \notin B) \wedge (x \notin C).$$

   Also, we have

   $$x \in (A - B) - C \iff x \in (A - B) \wedge x \notin C \iff (x \in A \wedge x \notin B) \wedge x \notin C.$$

   The statements on the right-hand side are equivalent. Hence $A - (B \cup C) = (A - B) - C$. □

   Proof Option 2.

   $$\begin{aligned} A - (B \cup C) &= A \cap (B \cup C)^c && \text{by the Set Difference Law} \\ &= A \cap (B^c \cap C^c) && \text{by De Morgan's law} \\ &= (A \cap B^c) \cap C^c && \text{by Associative law} \\ &= (A - B) \cap C^c && \text{by the Set Difference Law} \\ &= (A - B) - C && \text{by the Set Difference Law} \quad \square \end{aligned}$$

§24 — Functions Defined on General Sets

Epp 4th ed. pp. 383–396 · 5th ed. (metric) pp. 425–439. Lecture 18 (week 7).

Problems

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 1$.
   (a) What is the domain, codomain and range of f?
   (b) What is the image of $\{x \in \mathbb{R} \mid -1 \leq x \leq 2\}$ under f?
   (c) What is the preimage of $\{y \in \mathbb{R} \mid 0 \leq y \leq 3\}$ under f?
2. Let $X = \{1, 3, 5\}$ and $Y = \{s, t, u, v\}$ be sets. Define $f : X \to Y$ by the arrow diagram

   $$1 \mapsto v, \qquad 3 \mapsto s, \qquad 5 \mapsto v$$

   (the elements t and u of Y receive no arrows).

(a) Write the domain of f and the co-domain of f.

(b) Determine $f(1)$, $f(3)$, and $f(5)$.

(c) Determine the range of f.

(d) Is 3 an inverse image of s? Is 1 an inverse image of u?

(e) What is the inverse image of s? of u? of v?

(f) Represent f as a set of ordered pairs.

3. Let D be the set of all finite subsets of $\mathbb{Z}^+$. Consider the function $T : \mathbb{Z}^+ \to D$ defined by $T(n) := \{d \mid d \in \mathbb{Z}^+ \text{ and } d \text{ divides } n\}$. Find the following:

   (a) $T(1)$ (b) $T(15)$ (c) $T(17)$ (d) $T(5)$ (e) $T(18)$ (f) $T(21)$

4. Let S be the set of all strings of a's and b's.

   (a) Define the function $f : S \to \mathbb{Z}$ by

$$f(s) = \begin{cases} \text{the number of } b\text{'s to the left of the left-most } a \text{ in } s, & \text{if } s \text{ contains an } a, \\ 0, & \text{if } s \text{ contains no } a\text{'s.} \end{cases}$$

   Find $f(aba)$, $f(bbab)$, and $f(b)$. What is the range of f?

   (b) Define the function $g : S \to S$ by

$$g(s) = \text{the string obtained by writing the characters of } s \text{ in reverse order.}$$

   Find $g(aba)$, $g(bbab)$, and $g(b)$. What is the range of g?

Solutions

1. (a) The domain is $\mathbb{R}$, the codomain is $\mathbb{R}$, and the range is $[-1, \infty)$.

   (b) $f(\{x \in \mathbb{R} \mid -1 \leq x \leq 2\}) = [-1, 3]$.

   (c) $f^{-1}(\{y \in \mathbb{R} \mid 0 \leq y \leq 3\}) = \{x \in \mathbb{R} \mid 1 \leq x^2 \leq 4\} = [-2, -1] \cup [1, 2]$.

2. (a) The domain is $X = \{1, 3, 5\}$ and the codomain is $Y = \{s, t, u, v\}$.

   (b) $f(1) = v$, $f(3) = s$, $f(5) = v$.

   (c) $\text{range}(f) = \{s, v\}$.

   (d) **Yes**, 3 is an inverse image of s. **No**, 1 is not an inverse image of u.

(e) $f^{-1}(s) = \{3\}$, $f^{-1}(u) = \emptyset$, $f^{-1}(v) = \{1, 5\}$.

(f) $f = \{(1, v), (3, s), (5, v)\}$.

3. (a) $T(1) = \{1\}$.

(b) $T(15) = \{1, 3, 5, 15\}$.

(c) $T(17) = \{1, 17\}$.

(d) $T(5) = \{1, 5\}$.

(e) $T(18) = \{1, 2, 3, 6, 9, 18\}$.

(f) $T(21) = \{1, 3, 7, 21\}$.

4. (a) $f(aba) = 0$, $f(bbab) = 2$, $f(b) = 0$, $\text{range}(f) = \mathbb{Z}^{\geq 0}$.

(b) $g(aba) = aba$, $g(bbab) = babb$, $g(b) = b$, $\text{range}(g) = S$.

§25 — one-to-one, Onto, and Inverse Functions

Epp 4th ed. pp. 397–416 · 5th ed. (metric) pp. 439–461. Lecture 19 (week 7).

Problems

1. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = \left\lfloor \dfrac{1 - 6n}{3} \right\rfloor$.

(a) Is f one-to-one?

(b) Is f onto?

(c) Is f bijective?

(d) Prove or disprove that for all $m, n \in \mathbb{Z}$, we have $f(mn) = f(m)f(n)$.

2. Define the set $A = \{0, 1, 2, 3, 4\}$ and the functions $f : A \to A$ and $g : A \to A$ by

$$f(a) = (a + 4)^2 \pmod 5, \qquad g(a) = (a^2 + 3a + 1) \pmod 5.$$

Is $f = g$?

3. Consider the function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f((m, n)) = 2^m 3^n$.

(a) Determine $f((3, 1))$.

(b) Determine the pre-image of $\{36\}$.

(c) Prove f is one-to-one.

4. Consider the function $f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = \lfloor x^2 \rfloor$.

   (a) What is the image $f([-1, 1])$?

   (b) What is the preimage $f^{-1}(\{1\})$?

   (c) Is f one-to-one?

   (d) Is f onto?

   (e) Is f bijective?

Solutions

1. First, we note that, for $n \in \mathbb{Z}$,

$$f(n) = \left\lfloor \frac{1 - 6n}{3} \right\rfloor = \left\lfloor -2n + \frac{1}{3} \right\rfloor = -2n$$

since $-2n \in \mathbb{Z}$ and $\frac{1}{3} \in [0, 1)$.

   (a) **Yes.** If $f(m) = f(n)$ then $-2m = -2n$, hence $m = n$. Therefore f is one-to-one.

   (b) **No.** Since $f(n) = -2n$ we see that f never maps to any odd integer; for example, there is no integer w with $f(w) = 1$. Therefore f is not onto.

   (c) **No**, f is not a bijection. (It is injective but not surjective.)

   (d) This is **false**; take $m = n = 1$. Then $f(mn) = -2$ but $f(m)f(n) = 4$.

2. **Yes.** First note f and g have the same domain and codomain. This means we just need to show $f(a) = g(a)$ for all $a \in A$. Notice that we have

$$(a + 4)^2 = a^2 + 8a + 16 \equiv a^2 + 3a + 1 \pmod 5,$$

   so $f(a) = g(a)$ for all $a \in A$, and hence $f = g$.

3. (a) $f((3, 1)) = 2^3 3^1 = 24$.

   (b) $36 = 2^2 3^2$, so $f((m, n)) = 36$ if and only if $(m, n) = (2, 2)$. Hence $f^{-1}(\{36\}) = \{(2, 2)\}$.

   (c) If $f((m, n)) = f((m', n'))$, then $2^m 3^n = 2^{m'} 3^{n'}$, so $2^{m-m'} = 3^{n'-n}$. By unique prime factorisation, this forces $m - m' = 0$ and $n' - n = 0$. That is, $m = m'$ and $n = n'$, and hence $(m, n) = (m', n')$. So f is one-to-one.

4. (a) Note that if $x \in [-1, 1]$ then $x^2 \in [0, 1]$. This means we have $\lfloor x^2 \rfloor \in \{0, 1\}$. Both values occur because $\lfloor 0^2 \rfloor = 0$ and $\lfloor 1^2 \rfloor = 1$. Thus $f([-1, 1]) = \{0, 1\}$.

(b) $f^{-1}(\{1\}) = \{x \in \mathbb{R} : 1 \leq x^2 < 2\} = (-\sqrt{2}, -1] \cup [1, \sqrt{2})$.

(c) **No**, f is not one-to-one. For example, $f\left(\frac{1}{2}\right) = 0 = f\left(-\frac{1}{2}\right)$ but $\frac{1}{2} \neq -\frac{1}{2}$.

(d) **No**, f is not onto since $f(x) = \lfloor x^2 \rfloor \geq 0$ for all x. This means, for instance, that -1 is not in the image of f.

(e) **No**, f is not bijective.

§26 — Composition of Functions

Epp 4th ed. pp. 416–427 · 5th ed. (metric) pp. 461–472. Lecture 20 (week 7).

Problems

1. Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^3 - 1$ and $g(x) = x + 5$.
   (a) Determine the functions $(g \circ f)(x)$ and $(f \circ g)(x)$.
   (b) Is $g \circ f$ a bijection?
2. Define $f : \mathbb{Q} \to \mathbb{Z}$ by $f(x) = 2\lfloor x + 1 \rfloor$.
   (a) What is the range of f?
   (b) Is f one-to-one?
   (c) Is f onto?
   (d) Define $g : \mathbb{Z} \to \mathbb{Z}$ by $g(x) = 2x - 1$. Determine the composition $(g \circ f)(x)$.
   (e) Calculate $(g \circ f)\left(-\frac{3}{4}\right)$.
   (f) What is the range of $g \circ f$?
3. Let f and g be functions. Prove or disprove the following statements:
   (a) If f and $f \circ g$ are one-to-one, then g is one-to-one.
   (b) If f and $f \circ g$ are onto, then g is onto.

Solutions

1. (a) We have

$$(g \circ f)(x) = g(x^3 - 1) = x^3 + 4$$

and

$$(f \circ g)(x) = f(x + 5) = (x + 5)^3 - 1 = x^3 + 15x^2 + 75x + 124.$$

(b) **No**, $g \circ f$ is not a bijection. The map $x \mapsto x^3 + 4$ is one-to-one on $\mathbb{Z}$, but it is not onto (e.g. 0 is not in its image since $x^3 = -4$ has no integer solution).

2. (a) The range is a subset of the even integers because $\lfloor x + 1 \rfloor \in \mathbb{Z}$. If $2m$ is an even integer then $f(m - 1) = 2\lfloor m - 1 + 1 \rfloor = 2m$, so the range is the set of all even integers.

(b) **No**, f is not one-to-one. For example,

$$f(0) = 2\lfloor 0 + 1 \rfloor = 2\lfloor 1 \rfloor = 2 \times 1 = 2, \quad \text{and}$$
$$f\left(\tfrac{1}{2}\right) = 2\left\lfloor \tfrac{1}{2} + 1 \right\rfloor = 2\left\lfloor \tfrac{3}{2} \right\rfloor = 2 \times 1 = 2,$$

but $0 \neq \frac{1}{2}$.

(c) **No**, f is not onto since $\text{range}(f) \neq \mathbb{Z}$ (e.g. there is no $x \in \mathbb{Q}$ such that $f(x) = 1$ because 1 is odd).

(d) We have

$$(g \circ f)(x) = g(f(x)) = 2(2\lfloor x + 1 \rfloor) - 1 = 4\lfloor x + 1 \rfloor - 1.$$

(e) We have

$$(g \circ f)\left(-\frac{3}{4}\right) = 4 \cdot 0 - 1 = -1.$$

(f) The range is a subset of $\{4n - 1 \mid n \in \mathbb{Z}\}$ because $\lfloor x + 1 \rfloor \in \mathbb{Z}$. This *is* the range because, given an integer $4n - 1$, we have $f(n - 1) = 4\lfloor n - 1 + 1 \rfloor - 1 = 4n - 1$. Note that you can also write the range as $\{n \in \mathbb{Z} \mid n \equiv -1 \pmod 4\}$ which is the same as $\{n \in \mathbb{Z} \mid n \equiv 3 \pmod 4\}$.

3. (a) The statement is **true**.

Proof. Suppose that f and $f \circ g$ are one-to-one functions. Suppose for a contradiction that g is not one-to-one. Thus there exist two distinct elements in the domain of g, say x_1 and x_2, for which $g(x_1) = g(x_2)$. Let $g(x_1) = y = g(x_2)$. Thus $f(g(x_1)) = f(y) = f(g(x_2))$, but $x_1 \neq x_2$, which contradicts the fact that $f \circ g$ is one-to-one. Therefore g must be one-to-one. □

(b) The statement is **false**.

Counterexample. Consider the functions $g : \{1\} \to \{2, 3\}$ defined by $g(1) = 2$, and $f : \{2, 3\} \to \{4\}$ defined by $f(2) = 4$ and $f(3) = 4$. Then f is onto, and $f \circ g$ is onto (since $f \circ g : \{1\} \to \{4\}$ and $f(g(1)) = 4$). However, g is not onto. □

§27 — Cardinalities

Epp 4th ed. pp. 428–441 · 5th ed. (metric) pp. 473–486. Lecture 21 (week 8).

Problems

1. Let $A = \{-2, -1, 0, 1\}$ and $B = \{3, 4, 5, 6\}$. Prove that $|A| = |B|$ by providing a bijection between A and B.
2. Prove that $|\{1, 2, 3\}| \leq |\mathbb{Z}|$ by providing an appropriate injective (one-to-one) function.
3. Consider the intervals

$$[6, 12] = \{x \in \mathbb{R} \mid 6 \leq x \leq 12\} \quad \text{and} \quad (6, 12) = \{x \in \mathbb{R} \mid 6 < x < 12\}.$$

   Prove that $|[6, 12]| \leq |(6, 12)|$ by providing an appropriate injective (one-to-one) function.
4. Let $A = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$ and $B = \{x \in \mathbb{R} \mid 3 \leq x \leq 7\}$. Prove that $|A| = |B|$ by providing a bijection between A and B.

Solutions

1. Define the function $f : A \to B$ by

$$f(-2) = 3, \qquad f(-1) = 4, \qquad f(0) = 5, \qquad f(1) = 6.$$

   Then f is bijective, hence $|A| = |B|$.
2. Define the function $g : \{1, 2, 3\} \to \mathbb{Z}$ by $g(1) = 1$, $g(2) = 2$, and $g(3) = 3$. Then g is injective (one-to-one), so $|\{1, 2, 3\}| \leq |\mathbb{Z}|$.

3. Define $h : [6, 12] \to (6, 12)$ by

$$h(x) = \frac{x}{2} + 4.$$

For each $x \in [6, 12]$ we have

$$\begin{aligned} \frac{6}{2} + 4 \leq h(x) &\leq \frac{12}{2} + 4 \\ 7 \leq h(x) &\leq 10. \end{aligned}$$

Thus h is indeed a function from $[6, 12]$ to $(6, 12)$, and h has range $[7, 10] \subseteq (6, 12)$.

Suppose $x, y \in [6, 12]$ and $h(x) = h(y)$. Then $\frac{x}{2} + 4 = \frac{y}{2} + 4$, so $x = y$. Therefore h is injective (one-to-one).

Thus $|[6, 12]| \leq |(6, 12)|$.

4. Let $A = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$ and $B = \{x \in \mathbb{R} \mid 3 \leq x \leq 7\}$.

Consider the function $f : A \to B$ where $f(x) = 2x + 5$. Since $x \geq -1$, $f(x) = 2x + 5 \geq 2(-1) + 5 = 3$, and since $x \leq 1$, $f(x) = 2x + 5 \leq 2(1) + 5 = 7$. Also each value of x gives a unique value of $2x + 5$. Thus f is a function with the correct domain and codomain.

We now prove that f is one-to-one. Suppose that $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$. Thus $2x_1 + 5 = 2x_2 + 5$, and so $x_1 = x_2$. Thus f is one-to-one.

We now prove that f is onto. Suppose $y \in [3, 7]$. Let $x = \frac{y-5}{2}$. Since $y \leq 7$, we have $x \leq 1$ and since $y \geq 3$, we have $x \geq -1$. So $x \in [-1, 1]$. Furthermore $f(x) = f\left(\frac{y-5}{2}\right) = 2\left(\frac{y-5}{2}\right) + 5 = y - 5 + 5 = y$. Thus f is onto.

Hence f is a bijection and so $|A| = |B|$.

§28 — Countable and Uncountable Sets

Epp 4th ed. pp. 428–441 · 5th ed. (metric) pp. 473–486. Lecture 22 (week 8).

Note that this content will not be assessed in MATH1061.

Problems

1. Determine whether each of the following sets is countable or uncountable.
   (a) $\{1, 2, \ldots, n\}$.
   (b) $\mathbb{Z} \times \mathbb{Z}$.
   (c) $\mathbb{Q} \times \mathbb{Q}$.
   (d) $\mathbb{Z} \times \mathbb{Z} \times \cdots$ (infinitely many copies of $\mathbb{Z}$).
2. (a) Is $|\mathbb{Z}| = |\mathbb{Q}|$?
   (b) Is $|\mathbb{Z}| = |\mathbb{Z} \times \mathbb{Z}|$?
   (c) Is $|\mathbb{R}| = |\mathbb{Q} \times \mathbb{Q}|$?
3. (a) Show that the union of a countable number of countable sets is countable.
   (b) Show that the set of all finite binary strings is countable.
   (c) Use (a) to show that the set of all irrational numbers is uncountable.

Solutions

1. (a) The set is **countable** because the function $f : \{1, 2, \ldots, n\} \to \mathbb{N}$ defined by $f(i) = i$ is injective.
   (b) The set $\mathbb{Z} \times \mathbb{Z}$ is **countable**. Define the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ by

$$f(a,b) = \begin{cases} 2^a 3^b, & a > 0,\ b > 0, \\ 5^{-a} 7^{-b}, & a < 0,\ b < 0, \\ 11^a 13^{-b}, & a > 0,\ b < 0, \\ 17^{-a} 19^b, & a < 0,\ b > 0, \\ 23^b, & a = 0,\ b > 0, \\ 29^a, & a > 0,\ b = 0, \\ 31^{-b}, & a = 0,\ b < 0, \\ 37^{-a}, & a < 0,\ b = 0, \\ 1, & (a,b) = (0,0). \end{cases}$$

   We claim f is injective. Suppose $f(a,b) = f(c,d)$. By unique prime factorisation, the set of primes dividing $f(a,b)$ and the set of primes dividing $f(c,d)$ must be equal. Hence (a,b) and (c,d) lie in the same case. Again, by unique prime factorisation, the corresponding exponents must agree, so $a = c$ and $b = d$. Therefore f is injective.

(c) The set $\mathbb{Q} \times \mathbb{Q}$ is **countable**. First define an injection $e : \mathbb{Z} \to \mathbb{N}$ by

$$e(z) = \begin{cases} 2z, & z \geq 0, \\ -2z - 1, & z < 0. \end{cases}$$

Next define an injection $c : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$c(m, n) = 2^m 3^n.$$

Write each rational in lowest terms with positive denominator: $x = \frac{p}{q}$, $y = \frac{r}{s}$ with $p, r \in \mathbb{Z}$ and $q, s \in \mathbb{N}$. Define $F : \mathbb{Q} \times \mathbb{Q} \to \mathbb{N}$ by

$$F\left(\frac{p}{q}, \frac{r}{s}\right) = 2^{\,c(e(p),\, q)}\, 3^{\,c(e(r),\, s)}.$$

If $F\left(\frac{p}{q}, \frac{r}{s}\right) = F\left(\frac{p'}{q'}, \frac{r'}{s'}\right)$, then by unique prime factorisation we get $c(e(p), q) = c(e(p'), q')$ and $c(e(r), s) = c(e(r'), s')$. Since c is injective, we have $e(p) = e(p')$, $q = q'$, $e(r) = e(r')$, $s = s'$. Moreover, since e is injective, we have $p = p'$ and $r = r'$. Therefore $\frac{p}{q} = \frac{p'}{q'}$ and $\frac{r}{s} = \frac{r'}{s'}$, so F is injective.

(d) The set $\mathbb{Z} \times \mathbb{Z} \times ...$ is **not countable**. We will prove this by giving an injection $\mathbb{R} \to \mathbb{Z} \times \mathbb{Z} \times \cdots$.

For each $x \in \mathbb{R}$, write $x = n + r$ with $n = \lfloor x \rfloor \in \mathbb{Z}$ and $r \in [0, 1)$. Choose the unique decimal expansion of r that does not end with an infinite tail of 9's. Write this decimal expansion as

$$r = 0.d_1 d_2 d_3 \cdots, \qquad d_k \in \{0, 1, \ldots, 9\}.$$

Then define the function $\iota : \mathbb{R} \to \mathbb{Z} \times \mathbb{Z} \times \cdots$ by

$$\iota(x) = (n, d_1, d_2, d_3, \ldots) \in \mathbb{Z} \times \mathbb{Z} \times \cdots.$$

This is injective; if $\iota(x) = \iota(y)$ then the decimals of x and y are equal, so $x = y$.

2. (a) **Yes.** Both $\mathbb{Z}$ and $\mathbb{Q}$ are countable and infinite.

   (b) **Yes.** Both $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are countable and infinite.

   (c) **No.** $\mathbb{Q} \times \mathbb{Q}$ is countable, while $\mathbb{R}$ is uncountable.

3. (a) Let $A_1, A_2, A_3, ...$ be countable. Then for each n we can list A_n as

$$A_n = \{a_{n,1}, a_{n,2}, a_{n,3}, ...\}.$$

Now list the union by going along diagonals:

$$a_{1,1};\ a_{1,2}, a_{2,1};\ a_{1,3}, a_{2,2}, a_{3,1};\ a_{1,4}, a_{2,3}, a_{3,2}, a_{4,1};\ ...$$

Every element of every A_n appears somewhere in this list, so $\bigcup_{n=1}^{\infty} A_n$ can be listed in a sequence. Hence $\bigcup_{n=1}^{\infty} A_n$ is countable.

(b) Let B_n be the set of all binary strings of length n. Each B_n is finite (there are 2^n such strings), hence countable. Every finite binary string has some length, so

$$B = \{\text{all finite binary strings}\} = \bigcup_{n=0}^{\infty} B_n.$$

By (a), a countable union of countable sets is countable, so B is countable.

(c) Suppose, for contradiction, that the set of irrationals $\mathbb{R} \setminus \mathbb{Q}$ were countable. We already know $\mathbb{Q}$ is countable. Then by part (a),

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$$

would be a union of two countable sets, hence countable. But $\mathbb{R}$ is uncountable. This contradiction shows that $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

Application — Computability of Functions

Not assessable for MATH1061. Taken from K. Rosen, Discrete Mathematics and its Applications, *7th edition, 1991, pages 170–177.*

An important application of the topic of countability to computer science is examining the **computability** of functions. In computer science, a function is **computable** if there is a computer program in some programming language that can find the values of the function. If there is no such program, we say the function is **uncomputable**.

Application problems

1. (a) Show the set of all computer programs in any given computer language is countable.

   Hint: a computer program written in a programming language can be thought of as a string of symbols from a finite alphabet.

   (b) Let T be the set of all functions from $\mathbb{Z}^+$ to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Show that T is uncountable.

   Hint: start by finding an injection between the interval $(0, 1)$ *and a subset of* T*.*

   (c) Hence show that there are uncomputable functions.

Application solutions

1. (a) Given a computer language with a finite alphabet A, we let P be the set of all computer programs in the language. If P is finite, then it is countable and we are done, so assume P is infinite.

   Now we use the fact that we can think of each computer program as a series of strings made using the alphabet A. Consider all strings of length m made from alphabet A, and denote this set by A_m. It is easy to show that there are $|A|^m$ distinct strings of this type, and since A is finite, $|A|^m$ is also finite. Now P is a subset of $\bigcup_{i=1}^{\infty} A_m$. Since the union of a countable number of countable sets is also countable (see Question 3(a) from earlier in this section) it follows that $\bigcup_{i=1}^{\infty} A_m$ is countable and hence P is countable.

   (b) We use the hint and start by defining a function from $(0, 1)$ to T as follows. Recall that every real number in the interval $(0, 1)$ can be written in decimal notation as $0.a_1a_2a_3 \ldots a_n \ldots$ where each a_i is an integer from 0 to 9. Also recall that T is the set of all functions from $\mathbb{Z}^+$ to $\{0, 1, \ldots, 9\}$. Let $f : (0, 1) \to T$ be the function defined as

   $$f(0.a_1a_2a_3 \ldots a_n \ldots) = g,$$

   where g is the function in T such that $g(n) = a_n$ for all $n \in \mathbb{Z}^+$.

   So for example, if $f(0.457013) = g$ then g is the function where $g(1) = 4$, $g(2) = 5$, $g(3) = 7$, $g(4) = 0$, $g(5) = 1$, $g(6) = 3$ and $g(n) = 0$ for all $n \geq 7$.

   Now we will prove that this function is one-to-one. Suppose that $f(x_1) = f(x_2)$ for $x_1, x_2 \in (0, 1)$. Then for these functions to be equal, they map every integer to the same value. For this to happen, every decimal digit of x_1 must be equal to the same decimal digit of x_2 and so we must have $x_1 = x_2$.

   Now denote the set of all outputs of f as S. Hence $S \subseteq T$. Now if we set the codomain of f to be S, which is the range of f, we have made this function surjective and hence a

bijection. This means that the subset $S \subseteq T$ is in bijection with the interval $(0, 1)$. Since the interval $(0, 1)$ is uncountable, we must have that S is also uncountable and since $S \subseteq T$, T must also be uncountable.

(c) Part (a) showed us that the set of all computer programs in a given language is countable. This means that there are a countable number of computable functions. However part (b) showed us that there is an uncountable number of functions in the set T, so we must have that some functions are uncomputable.

§29 — Relations on Sets

Epp 4th ed. pp. 442–449 · 5th ed. (metric) pp. 487–494. Lecture 23 (week 8).

Problems

1. Define a relation P on $\mathbb{Z}$ as follows: for every ordered pair $(m, n) \in \mathbb{Z} \times \mathbb{Z}$,

$$m \; P \; n \iff m \text{ and } n \text{ have a common prime factor.}$$

   (a) Is $15 \; P \; 25$?

   (b) Is $22 \; P \; 27$?

   (c) Is $0 \; P \; 5$?

   (d) Is $8 \; P \; 8$?

2. Let $X = \{a, b, c\}$. Define a relation S on the power set $\mathcal{P}(X)$ as follows: for all sets A and B in $\mathcal{P}(X)$,

$$A \; S \; B \iff A \text{ has the same number of elements as } B.$$

   (a) Is $\{a, b\} \; S \; \{b, c\}$?

   (b) Is $\{a\} \; S \; \{a, b\}$?

   (c) Is $\{c\} \; S \; \{b\}$?

3. Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ be sets. Let R be the 'divides' relation. That is, for every ordered pair (a, b) in $A \times B$, we have $a \; R \; b$ if and only if $a \mid b$. Draw an arrow diagram (i.e. the directed graph) of this relation.

Solutions

1. (a) **Yes**, since $15 = 3 \cdot 5$ and $25 = 5 \cdot 5$ share the prime factor 5.

   (b) **No**, since $22 = 2 \cdot 11$ and $27 = 3^3$ share no common prime factor.

   (c) **Yes**, since 0 is divisible by every prime; in particular $5 \mid 0$ and $5 \mid 5$, so 0 and 5 have a common prime factor.

   (d) **Yes**, since $8 = 2^3$ and 2 is a prime factor of 8 (so they share the prime factor 2).

2. (a) **Yes**, since $|\{a, b\}| = 2 = |\{b, c\}|$.

   (b) **No**, since $|\{a\}| = 1 \neq 2 = |\{a, b\}|$.

   (c) **Yes**, since $|\{c\}| = 1 = |\{b\}|$.

3. We have $A \times B = \{(3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,4), (5,5), (5,6)\}$. The pairs (a, b) satisfying $a \mid b$ are

$$\{(3,6),\ (4,4),\ (5,5)\} \subseteq A \times B.$$

   So the directed graph has an arrow from 3 (in A) to 6 (in B), an arrow from 4 to 4, and an arrow from 5 to 5 — and no other arrows.

§30 — Reflexivity, Symmetry, Transitivity

Epp 4th ed. pp. 449–459 · 5th ed. (metric) pp. 495–505. Lecture 23 (week 8).

Problems

1. Let $X = \{1, 2, 3, 4, 5\}$ and consider the subset $R \subseteq X \times X$ given by

$$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,3), (3,4), (4,5), (5,1)\}.$$

   (a) Draw an arrow diagram for the binary relation R.

   (b) Draw an arrow diagram for the binary relation R^{-1}.

   (c) Draw an arrow diagram for the binary relation $R \cup R^{-1}$.

   (d) Is R symmetric? Is R^{-1} symmetric? Is $R \cup R^{-1}$ symmetric?

   (e) Is R transitive? Is R^{-1} transitive? Is $R \cup R^{-1}$ transitive?

2. Suppose R is a binary relation on the set $\{1, 2, 3\}$ such that:

- $(1, 2) \in R$
- $(2, 3) \in R$
- R is reflexive
- R is transitive

What is the minimum number of arrows required in an arrow diagram for a relation R that satisfies these criteria?

Solutions

1. (a) The arrow diagram for R has the five vertices $1, 2, 3, 4, 5$ drawn around a circle, with a **loop at every vertex** (from the pairs $(1, 1), \ldots, (5, 5)$) together with the directed 5-cycle

$$1 \to 2 \to 3 \to 4 \to 5 \to 1.$$

(b) The arrow diagram for R^{-1} is the same picture with every non-loop arrow reversed: a loop at every vertex, together with the directed 5-cycle

$$1 \to 5 \to 4 \to 3 \to 2 \to 1,$$

that is, $R^{-1} = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (2, 1), (3, 2), (4, 3), (5, 4), (1, 5)\}$.

(c) The arrow diagram for $R \cup R^{-1}$ has a loop at every vertex and a **double-headed arrow** between each consecutive pair around the cycle — that is, both $(i, i + 1)$ and $(i + 1, i)$ for the cycle $1, 2, 3, 4, 5, 1$.

(d) R is **not** symmetric because $(1, 2) \in R$ but $(2, 1) \notin R$.

R^{-1} is **not** symmetric because $(2, 1) \in R^{-1}$ but $(1, 2) \notin R^{-1}$.

$R \cup R^{-1}$ **is** symmetric. If $(a, b) \in R \cup R^{-1}$ then $(a, b) \in R$ or $(a, b) \in R^{-1}$, which means $(b, a) \in R^{-1}$ or $(b, a) \in R$, and thus, in either case, $(b, a) \in R \cup R^{-1}$.

(e) R is **not** transitive because $(1, 2), (2, 3) \in R$ but $(1, 3) \notin R$.

R^{-1} is **not** transitive because $(3, 2), (2, 1) \in R^{-1}$ but $(3, 1) \notin R^{-1}$.

$R \cup R^{-1}$ is **not** transitive because $(1, 2), (2, 3) \in R \cup R^{-1}$ but $(1, 3) \notin R \cup R^{-1}$.

2. R is reflexive, so $(1,1),(2,2),(3,3) \in R$ — a loop on each point of the arrow diagram. We are also given $(1,2),(2,3) \in R$, and R is transitive, so $(1,3) \in R$. R is *not* required to be symmetric, so no further arrows are forced.

$$R = \{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$$

So the **minimum number of arrows required is 6**.

§31 — Equivalence Relations

Epp 4th ed. pp. 459–478 · 5th ed. (metric) pp. 505–523. Lecture 24 (week 9).

Problems

1. Define the binary relations α, β and γ on $\mathbb{Z}$ as follows:

$$\begin{aligned} x \,\alpha\, y &\iff x + y \text{ is an even integer,} \\ x \,\beta\, y &\iff x \text{ is a multiple of } y, \\ x \,\gamma\, y &\iff 3 \text{ is a factor of } x^2 - y^2. \end{aligned}$$

   (a) Fill in the table with 'Yes'/'No' to indicate the properties satisfied by each relation.

	α	β	γ
Reflexive			
Symmetric			
Transitive			
Equivalence relation			

   (b) For each equivalence relation, state its equivalence classes.

2. Let $A = \{a,b,c,d,e\}$ be a set and ρ be a binary relation defined by

$$\rho = \{(a,a),(a,b),(a,d),(b,b),(c,c),(d,d),(b,a),(d,a),(b,d),(d,b),(c,e),(e,c),(e,e)\}.$$

   For instance, $(a,b) \in \rho$ means $a \,\rho\, b$.

   (a) Draw the directed graph representing ρ.

   (b) Is ρ reflexive? Is ρ symmetric? Is ρ transitive?

(c) Is ρ an equivalence relation on A? If so, give its equivalence classes.

(d) What is the smallest subset τ of ρ which forms an equivalence relation on A?

(e) Give the equivalence classes for τ.

3. Consider a positive integer n and any set of strings S. Define a relation R_n on S by the condition $s\ R_n\ t$ if and only if $s = t$ or both s and t have at least n characters and the first n characters are the same. Show that R_n is an equivalence relation.

Solutions

1. (a)

	α	β	γ
Reflexive	Yes	Yes	Yes
Symmetric	Yes	No	Yes
Transitive	Yes	Yes	Yes
Equivalence relation	Yes	No	Yes

(b) The equivalence relations are α and γ.

For α, notice $x\ \alpha\ y \iff x + y$ is even $\iff x$ and y have the same parity. This means the equivalence classes are $\mathbb{Z}^{\text{even}}$ and $\mathbb{Z}^{\text{odd}}$.

For γ, notice $x\ \gamma\ y \iff 3 \mid (x^2 - y^2) \iff x^2 \equiv y^2 \pmod 3$. But modulo 3 the only possible square residues are

$$0^2 \equiv 0, \qquad 1^2 \equiv 1, \qquad 2^2 \equiv 1 \pmod 3.$$

Hence $x^2 \equiv 0 \pmod 3$ exactly when $3 \mid x$, and $x^2 \equiv 1 \pmod 3$ exactly when $3 \nmid x$. Therefore $x\ \gamma\ y$ iff either both x and y are multiples of 3, or neither is. So the equivalence classes are $\{n \in \mathbb{Z} : 3 \mid n\}$ and $\{n \in \mathbb{Z} : 3 \nmid n\}$. Note that another way to write the equivalence classes is $\{3n \mid n \in \mathbb{Z}\}$ and $\{3n + 1,\ 3n + 2 \mid n \in \mathbb{Z}\}$.

2. (a) A digraph for ρ (vertex set $A = \{a, b, c, d, e\}$) has a **loop at every vertex**, double-headed arrows between each pair of $\{a, b, d\}$ — that is between a and b, between a and d, and between b and d — and a double-headed arrow between c and e. There are no arrows between $\{a, b, d\}$ and $\{c, e\}$.

(b) ρ is **reflexive**: we have $(x, x) \in \rho$ for all $x \in A$ (loops on all vertices).

ρ is **symmetric**: whenever $(x, y) \in \rho$, we have $(y, x) \in \rho$ (no single-direction arrow between any two distinct vertices).

ρ is **transitive**: within $\{a, b, d\}$ all relations occur, within $\{c, e\}$ all relations occur, and there are no relations between $\{a, b, d\}$ and $\{c, e\}$. That is, whenever (x, y) and (y, z) are in ρ, then (x, z) is also in ρ (and $x = z$ is possible).

(c) **Yes**, ρ is an equivalence relation. Its equivalence classes are

$$[a] = [b] = [d] = \{a, b, d\}, \qquad [c] = [e] = \{c, e\}.$$

(d) An equivalence relation $\tau \subseteq \rho$ must be reflexive, so $(x, x) \in \tau$ for all $x \in A$. But notice if τ is the relation

$$\tau = \{(a, a), (b, b), (c, c), (d, d), (e, e)\}$$

then τ is an equivalence relation. Therefore this is the smallest relation which is an equivalence relation.

(e) The equivalence classes for τ are the singletons:

$$[a] = \{a\},\ [b] = \{b\},\ [c] = \{c\},\ [d] = \{d\},\ [e] = \{e\}.$$

3. Let S be any set of strings and fix $n \in \mathbb{Z}^+$.

   R_n **is reflexive:** for any $s \in S$, since $s = s$, by definition $s\ R_n\ s$.

   R_n **is symmetric:** let $s, t \in S$, and assume $s\ R_n\ t$. If $s = t$ then $t\ R_n\ s$; otherwise both s and t must have at least n characters and, since $s\ R_n\ t$, they must have the same first n characters, so $t\ R_n\ s$.

   R_n **is transitive:** let $s, t, u \in S$ and assume $s\ R_n\ t$ and $t\ R_n\ u$. Since $s\ R_n\ t$ we either have $s = t$, or s and t both have at least n characters and have the same first n characters. Similarly, since $t\ R_n\ u$, either $t = u$, or t and u both have at least n characters and have the same first n characters. Thus either $s = t = u$, or s and u both have at least n characters and they both share the first n characters with t (and hence each other). Thus $s\ R_n\ u$ and so R_n is transitive.

   Since R_n is reflexive, symmetric and transitive it is an equivalence relation. □

Application — Identifiers

Not assessable for MATH1061. Taken from K. Rosen, Discrete Mathematics and its Applications, *7th edition, 1991, pages 608–618.*

In the C programming language, an **identifier** is the name of a variable, a function or another type of entity.

Each identifier is a nonempty string of characters where each character is a lowercase or uppercase English letter, a digit, or an underscore.

There is no limit on the number of characters used in the identifier; however, for some compilers for some versions of C, there is a limit on the number of characters checked when two names are compared.

For example, Standard C compilers consider two identifiers the same when they agree in their first 31 characters. We see that two identifiers will be considered the same if and only if they are related under the relation R_n when $n = 31$ (defined in a previous question).

Application problems

1. Using R_{31} on the set of all possible identifiers, determine the equivalence classes of each of the following identifiers:

   (a) `file_1`

   (b) `Number_of_tropical_storms`

   (c) `Number_of_named_tropical_storms`

   (d) `Number_of_named_tropical_storms_in_the_Atlantic_in_2005`

Application solutions

1. Let I be the set of all possible identifiers (strings over letters, digits and underscores).

   (a) Since `file_1` has length $6 < 31$, its equivalence class is just

   $$[\texttt{file_1}]_{R_{31}} = \{\texttt{file_1}\}.$$

   (b) Since `Number_of_tropical_storms` has length $25 < 31$, its equivalence class is

   $$[\texttt{Number_of_tropical_storms}]_{R_{31}} = \{\texttt{Number_of_tropical_storms}\}.$$

(c) The identifier `Number_of_named_tropical_storms` has length exactly 31, so two identifiers are R_{31}-equivalent to it iff they have at least 31 characters and their first 31 characters equal this string. Hence

$$[\texttt{Number_of_named_tropical_storms}]_{R_{31}} = \left\{ t \in I : \begin{array}{l} |t| \geq 31 \text{ and the first 31 characters of} \\ t \text{ are } \texttt{Number_of_named_tropical_storms} \end{array} \right\}.$$

(d) The identifier `Number_of_named_tropical_storms_in_the_Atlantic_in_2005` begins with the same first 31 characters `Number_of_named_tropical_storms`, so it has the same R_{31}-equivalence class as in (c):

$$[\texttt{Number_of_named_tropical_storms_in_the_Atlantic_in_2005}]_{R_{31}} = \left\{ t \in I : \begin{array}{l} |t| \geq 31 \text{ and the first 3} \\ t \text{ are } \texttt{Number_of_name} \end{array} \right.$$

§32 — Partial Order Relations

Epp 4th ed. pp. 498–515 · 5th ed. (metric) pp. 546–563. Lecture 25 (week 9).

Problems

1. Let ρ be a relation on $\mathbb{Z}$ defined by $x \; \rho \; y$ if and only if $x^3 - 1 < y^3$.

   (a) Prove ρ is a partial order relation.

   (b) Is ρ a total order?

2. Let S be the set of binary strings of length 6. Define the **weight** of a binary string as the number of ones it contains. (For example, weight(110110) = 4.) Define a relation ρ on S by

   $$s \; \rho \; t \text{ if and only if } \text{weight}(s) \leq \text{weight}(t).$$

   (a) Prove that ρ is not an equivalence relation.

   (b) Prove that ρ is not a partial order relation.

3. Suppose R is a binary relation on the set $\{1, 2, 3\}$ such that:

   - $(2, 3) \in R$
   - $(3, 2) \in R$
   - $(1, 1) \notin R$

- R is transitive
- R is not symmetric
- R is not antisymmetric.

Draw an arrow diagram for the relation R.

Solutions

1. (a) To show that ρ is a partial order, we need to show that ρ is reflexive, antisymmetric and transitive.

   Reflexive: let $x \in \mathbb{Z}$. Then $x^3 - 1 < x^3$ and so $x \rho x$, so ρ is reflexive.

   Antisymmetric: suppose $x \rho y$ where $x \neq y$. (We must show that then $y \not\rho x$.) Now $x \rho y$ means that $x^3 - 1 < y^3$. And since $x \neq y$ we must have $x^3 < y^3$ and so $x^3 \leqslant y^3 - 1$. Hence $y^3 - 1 \not< x^3$ and so we do not have $y \rho x$. Therefore ρ is antisymmetric.

   Transitive: suppose that $x \rho y$ and $y \rho z$. Then we know that $x^3 - 1 < y^3$ and $y^3 - 1 < z^3$. So $x^3 - 1 \leqslant y^3 - 1 < z^3$, that is, $x^3 - 1 < z^3$ so that $x \rho z$, which means that ρ is transitive.

   Hence ρ is a partial order. $\square$

   (b) To check whether ρ is a total order, let x and y be any two distinct integers. Then $x^3 \neq y^3$ and so either $x^3 < y^3$ or else $y^3 < x^3$.

   If $x^3 < y^3$ then $x^3 - 1 < y^3$ and so $x \rho y$.

   If $y^3 < x^3$ then $y^3 - 1 < x^3$ and so $y \rho x$.

   So for any integers $x \neq y$, we have either $x \rho y$ or else $y \rho x$. So any two integers are comparable under ρ. Hence ρ **is** a total order.

2. (a) It is **not an equivalence relation** because it is not symmetric. For instance, let $s = 000000$ and $t = 111111$. Then $\text{weight}(s) = 0 \leq 6 = \text{weight}(t)$, so $(s, t) \in \rho$, but $(t, s) \notin \rho$.

   (b) It is **not a partial order** because it is not antisymmetric. For instance, let $s = 100000$ and $t = 010000$. Then $\text{weight}(s) = \text{weight}(t) = 1$, so $s \rho t$ and $t \rho s$, but $s \neq t$.

3. One relation on $\{1, 2, 3\}$ satisfying all conditions is

$$R = \{(1, 2), (1, 3), (2, 2), (3, 3), (2, 3), (3, 2)\}.$$

(It contains $(2,3)$ and $(3,2)$; it is transitive; it is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$; it is not antisymmetric since $2 \neq 3$ and $(2,3),(3,2) \in R$; and $(1,1) \notin R$.)

An arrow diagram for this R has vertices $1,2,3$ with: arrows $1 \to 2$ and $1 \to 3$; loops at 2 and at 3; and a double-headed arrow between 2 and 3. There is no loop at vertex 1.

§33 — Definitions and Examples of Groups

No relevant textbook pages. Lecture 26 (week 9).

Problems

1. Show $\mathbb{Z}_9 - \{0\} = \{1,2,3,\ldots,7,8\}$ is not a group under multiplication modulo 9.
2. Let G be the set of all odd integers. Define the binary operation $\star$ on G by

$$x \star y = x + y - 1.$$

   Show that $(G, \star)$ is an abelian group.
3. Show $\mathbb{Z}_9 - \{0,3,6\} = \{1,2,4,5,7,8\}$ is a group under multiplication modulo 9. Is the group abelian?
4. Define the group $H = \mathbb{Z}_5 - \{0\} = \{1,2,3,4\}$ under multiplication modulo 5. Write the Cayley table for H.
5. Define the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ under component-wise addition modulo 2. That is,

$$(a,b) + (c,d) = (a + c \bmod 2,\ b + d \bmod 2).$$

   Write the Cayley table for G.

Solutions

1. It is not a group under multiplication mod 9 because, for example, it is **not closed** ($3 \cdot 3 = 0 \notin \mathbb{Z}_9 - \{0\}$).

2. **Closure:** if x and y are odd, then $x+y$ is even, so $x+y-1$ is odd. Hence $x \star y \in G$.

   Associativity: for $x, y, z \in G$,

$$(x \star y) \star z = (x+y-1)+z-1 = x+y+z-2 = x+(y+z-1)-1 = x \star (y \star z).$$

   Commutativity: for $x, y \in G$,

$$x \star y = x+y-1 = y+x-1 = y \star x.$$

   Identity: we seek $e \in G$ such that $x \star e = x = e \star x$ for all x. Now the requirement $x \star e = x+e-1 = x$ implies that $e = 1$. Note 1 lies in G. Moreover, commutativity implies that $e \star x = x$ as well. Thus the identity is 1.

   Inverses: given $x \in G$, we want $y \in G$ with $x \star y = 1$. This means $x+y-1 = 1$, so $y = 2-x$. Since x is odd, $2-x$ is odd, hence $2-x \in G$. Thus the inverse of x is $x^{-1} = 2-x$.

   Therefore $(G, \star)$ is an abelian group. $\square$

3. **Closure:** if $a, b \in \{1,2,4,5,7,8\}$, then $3 \nmid a$ and $3 \nmid b$, hence $3 \nmid ab$, so $ab \not\equiv 0, 3, 6 \pmod 9$; thus $\{1,2,4,5,7,8\}$ is closed under multiplication modulo 9.

   Associativity and commutativity: normal multiplication between integers is associative and commutative, so multiplication modulo 9 in the set $\{1,2,4,5,7,8\}$ is also associative and commutative. This means the group **is abelian**.

   Identity: $1 \in G$ is the identity because $1 \cdot x \equiv x \pmod 9$.

   Inverse: every element has an inverse. We explicitly calculate the inverses:

$$\begin{aligned}
1^{-1} &= 1 \quad \text{because } 1 \cdot 1 \equiv 1 \pmod 9, \\
2^{-1} &= 5 \quad \text{because } 2 \cdot 5 = 10 \equiv 1 \pmod 9, \\
4^{-1} &= 7 \quad \text{because } 4 \cdot 7 = 28 \equiv 1 \pmod 9, \\
5^{-1} &= 2 \quad \text{because } 5 \cdot 2 = 10 \equiv 1 \pmod 9, \\
7^{-1} &= 4 \quad \text{because } 7 \cdot 4 = 28 \equiv 1 \pmod 9, \\
8^{-1} &= 8 \quad \text{because } 8 \cdot 8 = 64 \equiv 1 \pmod 9.
\end{aligned}$$

4. For $H = \mathbb{Z}_5 \setminus \{0\} = \{1,2,3,4\}$ under multiplication mod 5, the Cayley table is:

$\cdot$	1	2	3	4
1	1	2	3	4

$\cdot$	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

5. For $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$ under componentwise addition mod 2, the Cayley table is:

$+$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(\mathbf{0},\mathbf{0})$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(\mathbf{0},\mathbf{1})$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(\mathbf{1},\mathbf{0})$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(\mathbf{1},\mathbf{1})$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$

§34 — Elementary Properties of Groups

No relevant textbook pages. Lecture 27 (week 10).

Problems

1. Let $H = \{3n \mid n \in \mathbb{Z}\}$ be the set of multiples of 3. Prove H is a subgroup of $(\mathbb{Z}, +)$.
2. Fix elements g and h in a group G where e is the group identity. Prove the following statements:
   (a) If $g^{-1} = h$ then $h^{-1} = g$.
   (b) $(g^{-1})^{-1} = g$.
   (c) $(gh)^{-1} = h^{-1}g^{-1}$.
3. Prove $K = \{(0,0),(0,1),(2,0),(2,1)\}$ is a subgroup of $(\mathbb{Z}_4 \times \mathbb{Z}_2, +)$ where the group operation is componentwise addition:

$$(a,b) + (c,d) = (a + c \pmod 4,\ b + d \pmod 2).$$

4. Consider the group $(\mathbb{Z}_{11} - \{0\}, \cdot)$. Show that this group is cyclic.
   Hint: express every element of $\mathbb{Z}_{11} - \{0\}$ as a power of 2.

Solutions

1. First we note that $H \subseteq \mathbb{Z}$, and $H \neq \emptyset$.

   Closure: if $x, y \in H$ then $x = 3a$ and $y = 3b$ for some $a, b \in \mathbb{Z}$. Therefore $x + y = 3a + 3b = 3(a + b) \in H$.

   Identity: the identity of $(\mathbb{Z}, +)$ is 0, and since $0 = 3 \cdot 0$, we have $0 \in H$.

   Inverse: in $\mathbb{Z}$, the inverse of x is $-x$. In H, if $x \in H$ then $x = 3a$ for some $a \in \mathbb{Z}$. This means $-x = -3a = 3 \cdot (-a)$, and so $-x \in H$ also.

   Note that we do not need to check associativity for subgroups, since this property is automatically inherited from the parent group $(\mathbb{Z}, +)$. □

2. (a) Suppose $g^{-1} = h$. Now $gh = gg^{-1} = e$ and $hg = g^{-1}g = e$. Thus $h^{-1} = g$.

   (b) By definition, $gg^{-1} = e = g^{-1}g$, so $(g^{-1})^{-1} = g$.

   (c) Notice we have

   $$(gh)(h^{-1}g^{-1}) = ghh^{-1}g^{-1} = e$$

   and

   $$(h^{-1}g^{-1})(gh) = h^{-1}g^{-1}gh = e.$$

   This means $(gh)^{-1} = h^{-1}g^{-1}$.

3. We draw the Cayley table of $K = \{(0,0), (0,1), (2,0), (2,1)\}$ under componentwise addition in $\mathbb{Z}_4 \times \mathbb{Z}_2$:

$+$	$(0,0)$	$(0,1)$	$(2,0)$	$(2,1)$
$(\mathbf{0},\mathbf{0})$	$(0,0)$	$(0,1)$	$(2,0)$	$(2,1)$
$(\mathbf{0},\mathbf{1})$	$(0,1)$	$(0,0)$	$(2,1)$	$(2,0)$
$(\mathbf{2},\mathbf{0})$	$(2,0)$	$(2,1)$	$(0,0)$	$(0,1)$
$(\mathbf{2},\mathbf{1})$	$(2,1)$	$(2,0)$	$(0,1)$	$(0,0)$

   Notice the operation is closed in K and every element of K has an inverse in K. Therefore K is a subgroup. □

4. Note that 2 is a generator of $(\mathbb{Z}_{11} - \{0\}, \cdot)$, so it is cyclic:

   $$1 = 2^0,\ 2 = 2^1,\ 4 = 2^2,\ 8 = 2^3,\ 5 = 2^4,\ 10 = 2^5,\ 9 = 2^6,\ 7 = 2^7,\ 3 = 2^8,\ 6 = 2^9.$$

§35 — Group Isomorphisms

No relevant textbook pages. No lecture in the published schedule — the Week 10 row jumps from L27 (V34) to L28 (V36).

This content is optional and will not be assessed. The source document carries no practice problems or solutions for this section.

§36 — Definitions and Examples of Fields

No relevant textbook pages. Lecture 28 (week 10).

Problems

1. (a) Solve $3x + 4 = 2$ in the field $(\mathbb{Z}_5, +, \cdot)$.
   (b) Solve $8x - 9 = 6$ in the field $(\mathbb{Z}_{11}, +, \cdot)$.
   (c) Solve $4(1 - x) = 10$ in the field $(\mathbb{Z}_{13}, +, \cdot)$.
   (d) Solve $16(x - 6) = 15$ in the field $(\mathbb{Z}_{19}, +, \cdot)$.
2. Prove $\mathbb{Z}_{12}$ with addition modulo 12 and multiplication modulo 12 is not a field.
3. Define addition and multiplication on $\mathbb{Q} \times \mathbb{Q}$ by

$$(a, b) + (c, d) = (a + c,\ b + d)$$

   and

$$(a, b) \cdot (c, d) = (ac + 3bd,\ ad + bc)$$

   for all $(a, b), (c, d) \in \mathbb{Q} \times \mathbb{Q}$. Prove that $(\mathbb{Q} \times \mathbb{Q}, +, \cdot)$ is a field.

Solutions

1. (a)

$$\begin{aligned} 3x + 4 &= 2 \\ \rightarrow 3x &= -2 \end{aligned}$$

Now, since $3 \times 2 = 6 \equiv 1 \pmod 5$, 2 is the multiplicative inverse of 3, and so we multiply both sides by 2 and obtain $x = -4$. That is, $x = 1$.

Check: $3 \cdot 1 + 4 = 7 \equiv 2 \pmod 5$.

(b)

$$\begin{aligned} 8x - 9 &= 6 \\ \rightarrow 8x &= 15 = 4 \end{aligned}$$

Now, since $8 \times 7 = 56 \equiv 1 \pmod{11}$, 7 is the multiplicative inverse of 8, and so we multiply both sides by 7 and obtain $x = 28$. That is, $x = 6$.

Check: $8 \cdot 6 - 9 = 39 \equiv 6 \pmod{11}$.

(c) Since $4 \times 10 = 40 \equiv 1 \pmod{13}$, 10 is the multiplicative inverse of 4, and so we multiply both sides by 10 and obtain $1 - x = 100 = 9$. That is, $x = 1 - 9 = 14 - 9 = 5$.

Check: $4(1 - 5) = -16 \equiv 10 \pmod{13}$.

(d) Since $16 \times 6 = 96 \equiv 1 \pmod{19}$, 6 is the multiplicative inverse of 16, and so we multiply both sides by 6 and obtain $x - 6 = 90 = 14$. That is, $x = 20 = 1$.

Check: $16(1 - 6) = -80 \equiv 15 \pmod{19}$.

2. $\mathbb{Z}_{12}$ is not a field because 2 has no multiplicative inverse modulo 12. Indeed $2 \cdot k \equiv 1 \pmod{12}$ has no solution since $2k$ is always even.

3. We show that $(\mathbb{Q} \times \mathbb{Q}, +, \cdot)$ is a field in three steps.

Step 1 — $(\mathbb{Q} \times \mathbb{Q}, +)$ is an abelian group.

Closure: for all $(a, b), (c, d) \in \mathbb{Q} \times \mathbb{Q}$, we have $(a, b) + (c, d) = (a + c, b + d) \in \mathbb{Q} \times \mathbb{Q}$.

Associative: for all $(a, b), (c, d), (e, f) \in \mathbb{Q} \times \mathbb{Q}$ we have $\big((a, b) + (c, d)\big) + (e, f) = (a + c, b + d) + (e, f) = (a + c + e, b + d + f) = (a, b) + (c + e, d + f) = (a, b) + \big((c, d) + (e, f)\big)$.

Identity: for all $(a, b) \in \mathbb{Q} \times \mathbb{Q}$ we have $(a, b) + (0, 0) = (a + 0, b + 0) = (a, b) = (0 + a, 0 + b) = (0, 0) + (a, b)$.

Inverses: for all $(a,b) \in \mathbb{Q} \times \mathbb{Q}$ we have $(a,b) + (-a,-b) = (a-a, b-b) = (0,0)$ and $(-a,-b) + (a,b) = (-a+a, -b+b) = (0,0)$. So $(-a,-b)$ is the inverse of (a,b).

Commutativity: for all $(a,b),(c,d) \in \mathbb{Q} \times \mathbb{Q}$, we have $(a,b) + (c,d) = (a+c, b+d) = (c+a, d+b) = (c,d) + (a,b)$.

Thus $(\mathbb{Q} \times \mathbb{Q}, +)$ is an abelian group.

Step 2 — $((\mathbb{Q} \times \mathbb{Q}) - \{(0,0)\},\ \cdot)$ is an abelian group.

Closure: for all $(a,b),(c,d) \in (\mathbb{Q}\times\mathbb{Q}) - \{(0,0)\}$, we have $(a,b)\cdot(c,d) = (ac+3bd, ad+bc) \in \mathbb{Q} \times \mathbb{Q}$. So we only need to show that $(ac+3bd, ad+bc) \neq (0,0)$. For a contradiction, suppose $(ac+3bd, ad+bc) = (0,0)$. That is, $ac+3bd = 0$ and $ad+bc = 0$. Multiplying the first equation by d and the second by c we obtain $acd + 3bd^2 = 0$ and $acd + bc^2 = 0$, from which it follows that $3bd^2 = bc^2$.

Now, if $b = 0$, then $ac = 0$, $ad = 0$ and $a \neq 0$, and so $c = d = 0$; a contradiction. Thus $b \neq 0$ and we have $3d^2 = c^2$. This is a contradiction because if $d = 0$, then $c \neq 0$ and the equation does not hold, and if $d \neq 0$, then we have $3 = \frac{c^2}{d^2}$, which implies $\sqrt{3} = \frac{c}{d}$ is rational. Thus $(ac+3bd, ad+bc) \neq (0,0)$ and we have closure.

Associative: for all $(a,b),(c,d),(e,f) \in (\mathbb{Q} \times \mathbb{Q}) - \{(0,0)\}$ we have

$$\begin{aligned}(a,b)\cdot\big((c,d)\cdot(e,f)\big) &= (a,b)\cdot(ce+3df,\ cf+de)\\ &= (a(ce+3df)+3b(cf+de),\ a(cf+de)+b(ce+3df))\\ &= (ace+3adf+3bcf+3bde,\ acf+ade+bce+3bdf)\end{aligned}$$

and

$$\begin{aligned}\big((a,b)\cdot(c,d)\big)\cdot(e,f) &= (ac+3bd,\ ad+bc)\cdot(e,f)\\ &= \big((ac+3bd)e+3(ad+bc)f,\ (ac+3bd)f+(ad+bc)e\big)\\ &= (ace+3bde+3adf+3bcf,\ acf+3bdf+ade+bce)\\ &= (ace+3adf+3bcf+3bde,\ acf+ade+bce+3bdf).\end{aligned}$$

Identity: for all $(a,b) \in (\mathbb{Q} \times \mathbb{Q}) - \{(0,0)\}$ we have $(a,b)\cdot(1,0) = (a,b)$ and $(1,0)\cdot(a,b) = (a,b)$, so $(1,0)$ is a multiplicative identity.

Commutativity: for all $(a,b),(c,d) \in \mathbb{Q} \times \mathbb{Q}$, we have

$$(a,b)\cdot(c,d) = (ac+3bd,\ ad+bc) = (ca+3db,\ cb+da) = (c,d)\cdot(a,b).$$

Inverses: we check that for all $(a,b) \in (\mathbb{Q} \times \mathbb{Q}) - \{(0,0)\}$,

$$(a,b)^{-1} = \left(\frac{a}{a^2-3b^2},\ \frac{-b}{a^2-3b^2}\right).$$

First, note that $a^2-3b^2 \neq 0$, because $a^2-3b^2=0$ implies either $(a,b)=(0,0)$ or $\sqrt{3}=\frac{a}{b}$, both of which are contradictions (the second because $\sqrt{3}$ is not rational). Thus $(a,b)^{-1} \in (\mathbb{Q}\times\mathbb{Q})-\{(0,0)\}$, and notice

$$\begin{aligned}(a,b)\cdot\left(\frac{a}{a^2-3b^2},\frac{-b}{a^2-3b^2}\right) &= \left(a\cdot\frac{a}{a^2-3b^2}+3b\cdot\frac{-b}{a^2-3b^2},\ a\cdot\frac{-b}{a^2-3b^2}+b\cdot\frac{a}{a^2-3b^2}\right)\\ &= \left(\frac{a^2-3b^2}{a^2-3b^2},\ \frac{-ab+ab}{a^2-3b^2}\right)\\ &= (1,0).\end{aligned}$$

So every nonzero element has an inverse. Thus $((\mathbb{Q}\times\mathbb{Q})-\{(0,0)\},\cdot)$ is an abelian group.

Step 3 — the distributive laws. For all $(a,b),(c,d),(e,f)\in\mathbb{Q}\times\mathbb{Q}$ we have

$$\begin{aligned}(a,b)\cdot((c,d)+(e,f)) &= (a,b)\cdot(c+e,\ d+f)\\ &= (a(c+e)+3b(d+f),\ a(d+f)+b(c+e))\\ &= ((ac+3bd)+(ae+3bf),\ (ad+bc)+(af+be))\\ &= (ac+3bd,\ ad+bc)+(ae+3bf,\ af+be)\\ &= (a,b)\cdot(c,d)+(a,b)\cdot(e,f)\end{aligned}$$

and (since we have already shown that multiplication and addition are commutative)

$$\begin{aligned}((a,b)+(c,d))\cdot(e,f) &= (e,f)\cdot((a,b)+(c,d))\\ &= (e,f)\cdot(a,b)+(e,f)\cdot(c,d) \quad \text{(by the first distributive law)}\\ &= (a,b)\cdot(e,f)+(c,d)\cdot(e,f).\end{aligned}$$

Thus $(\mathbb{Q}\times\mathbb{Q},+,\cdot)$ is a field. □

§37 — Introduction to Counting

Epp 4th ed. pp. 516–553 · 5th ed. (metric) pp. 564–604. Lecture 29 (week 11).

Problems

1. The UQ drama club is auditioning for its annual production of *Romeo and Juliet*. Six people are auditioning for the role of Romeo, and eight other people are auditioning for the role of Juliet. In how many ways can the director cast the lead couple?
2. There are eleven questions on the final exam, but you decide — based on your successful grade for your in-semester exam — that you will only do five of them. How many combinations of questions are there which you could complete?
3. (a) How many permutations of $\{1, 2, \dots, 15\}$ start with an odd number?

   (b) How many 5-element subsets of $\{1, 2, \dots, 15\}$ contain exactly two numbers from $\{1, 2, 3, 4\}$?

Solutions

1. There are 6 choices for Romeo and 8 choices for Juliet, so $6 \cdot 8 = 48$ ways.
2. Since the order in which you do the 5 questions does not matter, there are $\binom{11}{5} = 462$ combinations of questions.
3. (a) There are 8 odd numbers in the set $\{1, \dots, 15\}$. Choose the first entry in 8 ways, then permute the remaining 14 numbers, so $8 \cdot 14!$ permutations.

   (b) We choose exactly 2 numbers from the set $\{1, 2, 3, 4\}$ and exactly 3 numbers from the remaining 11 numbers, so the number of subsets is

$$\binom{4}{2}\binom{11}{3}.$$

Application — Runtime

Not assessable for MATH1061. Taken from S. Epp, Discrete Mathematics with Applications, *5th edition, pages 787–799.*

In many algorithms, the runtime is well approximated by counting how many times the basic instructions are executed.

For programs with nested `for`-loops and no early exits, the total number of executions of the inner loop body equals the number of choices of the outer-loop index multiplied by the number of choices of the inner-loop index.

Concretely, if i runs through M values and, for each fixed i, the variable j runs through N values, then the inner loop body is executed exactly MN times.

Application problems

1. The following algorithms are written in pseudo-code and contain nested loops. For each algorithm, count the total number of iterations of the inner loop that the algorithm will run.

   (a)

```
for i := 1 to 30
    for j := 1 to 15
        [Statements in body of inner loop. None contain
         branching statements that lead outside the loop.]
    next j
next i
```

   (b)

```
for i := 5 to 50
    for j := 10 to 20
        [Statements in body of inner loop. None contain
         branching statements that lead outside the loop.]
    next j
next i
```

   (c) (Fix integers $a, b, c, d \in \mathbb{Z}^+$ with $a \leq b$ and $c \leq d$.)

```
for i := a to b
    for j := c to d
        [Statements in body of inner loop. None contain
         branching statements that lead outside the loop.]
    next j
next i
```

Application solutions

1. (a) $i = 1, \ldots, 30$ gives 30 iterations and $j = 1, \ldots, 15$ gives 15 iterations, so the inner loop runs $30 \cdot 15 = 450$ times.

   (b) $i = 5, \ldots, 50$ gives $50 - 5 + 1 = 46$ iterations and $j = 10, \ldots, 20$ gives $20 - 10 + 1 = 11$ iterations, so the inner loop runs $46 \cdot 11 = 506$ times.

   (c) $i = a, \ldots, b$ gives $b - a + 1$ iterations and $j = c, \ldots, d$ gives $d - c + 1$ iterations, so the inner loop runs $(b - a + 1)(d - c + 1)$ times.

§38 — Counting Selections

Epp 4th ed. pp. 516–553 · 5th ed. (metric) pp. 564–604. Lecture 30 (week 11).

Problems

1. (a) How many distinct arrangements can be made from the letters of the word INDOOROOPILLY?

   (b) How many distinct arrangements can be made from the letters of the word INDOOROOPILLY that begin with R and end with P?

   (c) How many distinct arrangements can be made from the letters of the word INDOOROOPILLY that begin with I and end with Y?

   (d) How many distinct arrangements can be made from the letters of the word INDOOROOPILLY that have all four Os next to each other?

2. (a) How many distinct arrangements can be made of the letters of the word WOOLLOONGABBA?

   (b) How many distinct arrangements are there of the letters of the word WOOLLOONGABBA such that **exactly one** of the following is true?

   - The two As are not next to each other.
   - The two Bs are not next to each other.
   - The two Ls are not next to each other.

3. (a) How many one-to-one (injective) functions are there from $\{a, b, c\}$ to $\{1, 2, 3, 4, 5\}$?

   (b) How many onto (surjective) functions are there from $\{1, 2, 3, 4, 5\}$ to $\{a, b, c\}$?

Solutions

1. (a) There are 13 letters in INDOOROOPILLY, with four O's, two I's, two L's, and the remaining letters appearing once each. Therefore the number of distinct arrangements is

$$\frac{13!}{4!\,2!\,2!} = 64\,864\,800.$$

   (b) Fix R in the first position and P in the last position. There are 11 letters remaining, with four O's, two I's, two L's, and the remaining letters appearing once each. Therefore the number of distinct arrangements is

$$\frac{11!}{4!\,2!\,2!} = 415\,800.$$

(c) Fix I in the first position and Y in the last position. There are 11 letters remaining, with four O's, two L's, and the remaining letters appearing once each. Therefore the number of distinct arrangements is

$$\frac{11!}{4!\,2!} = 831\,600.$$

(d) Treat the four O's as one 'letter'. There are 10 'letters' with two I's, two L's, and the remaining 'letters' appearing once each. Therefore the number of distinct arrangements is

$$\frac{10!}{2!\,2!} = 907\,200.$$

2. (a) There are 13 letters in WOOLLOONGABBA, with four O's, two L's, two A's, two B's, and the remaining letters appearing once each. Therefore the number of distinct arrangements is

$$\frac{13!}{4!\,2!\,2!\,2!} = 32\,432\,400.$$

(b) We want exactly one of the pairs AA, BB, LL to be non-adjacent. That is, there are three cases to consider:

- **Case 1:** the two A's are not adjacent but BB and LL are adjacent.
- **Case 2:** the two B's are not adjacent but AA and LL are adjacent.
- **Case 3:** the two L's are not adjacent but AA and BB are adjacent.

Note that the number of ways Case 1 can occur is exactly the same as the number of ways Case 2 can occur (and similarly for Case 3).

To count the number of ways Case 1 can occur, we first count arrangements with BB adjacent and LL adjacent. Treat BB as one 'letter' and LL as one 'letter'. Then there are 11 'letters' (W, O, O, LL, O, O, N, G, A, BB, A) to arrange, with four O's, two A's, and the remaining 'letters' appearing once each. Therefore the number of such arrangements is

$$\frac{11!}{4!\,2!}.$$

In order to determine the number of these for which the two A's are not adjacent, we take this total and subtract those where AA is also adjacent. For this, treat AA, BB, and LL as 'letters'. Then there are 10 'letters' to arrange, with four O's and the remaining 'letters' appearing once each. Therefore the number of arrangements with AA adjacent and BB adjacent and LL adjacent is

$$\frac{10!}{4!}.$$

Hence, the number of ways that Case 1 can occur is

$$\frac{11!}{4!\,2!} - \frac{10!}{4!}.$$

Case 2 and Case 3 have exactly the same number of occurrences as Case 1. Also note that the three cases are mutually exclusive (you cannot have both Case 1 and Case 2 occurring at the same time, for example).

So the number of arrangements of the word WOOLLOONGABBA for which exactly one of the three cases occurs is

$$3\left(\frac{11!}{4!\,2!} - \frac{10!}{4!}\right).$$

3. (a) An injective function $f : \{a, b, c\} \to \{1, 2, 3, 4, 5\}$ is determined by choosing distinct images for a, b, c. The number of choices is

$$5 \cdot 4 \cdot 3 = {}^5P_3 = 60.$$

(b) The number of surjective functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c\}$ is the total number of functions from $\{1, 2, 3, 4, 5\}$ to $\{a, b, c\}$ minus the number of functions from $\{1, 2, 3, 4, 5\}$ to $\{a, b, c\}$ which are not surjective. The former number is 3^5. A function which is not surjective must have image of size 1 or 2.

- In the first case, there are 3 such functions (one for each of the three choices for the image).
- In the latter case, first choose which 2 elements of $\{a, b, c\}$ will be in the image. There are $\binom{3}{2}$ such choices. For this fixed two-element image, the number of functions with this image is

$$2^5 - 2,$$

since there are 2^5 total functions from a 5-element set to a 2-element set, and we subtract the 2 constant ones (which fall into the first case, having image of size 1). Thus the number of functions from $\{1, 2, 3, 4, 5\}$ to $\{a, b, c\}$ with an image of size 2 is

$$\binom{3}{2}(2^5 - 2) = 3 \cdot 30 = 90.$$

This means there are 93 functions which are not surjective, so the number of surjective functions from $\{1, 2, 3, 4, 5\}$ to $\{a, b, c\}$ is

$$3^5 - 93 = 150.$$

§39 — Introduction to Probability

Epp 4th ed. pp. 516–553 · 5th ed. (metric) pp. 564–604. Lecture 31 (week 11).

Problems

1. To form a committee, five people are to be chosen from a group of ten people.
   (a) How many different committees of five can be chosen?
   (b) Amongst the ten people, we have Alice, Bob, Eve and Oscar. Alice and Bob are extremely talkative during meetings, and so at most one of these two can be chosen. Eve and Oscar are inseparable, and so a committee must either contain both of them or neither of them. With these constraints, how many different five-person committees can be formed?
   (c) What is the probability that none of Alice, Bob, Eve and Oscar are on a committee, given that a random committee of five is chosen subject to the constraints in part (b)?
2. (a) How many positive two-digit integers are multiples of 3?
   (b) What is the probability that a randomly chosen positive two-digit integer is a multiple of 3?
   (c) What is the probability that a randomly chosen positive two-digit integer is a multiple of 4?

Solutions

1. (a) Number of different committees of five is

$$\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2} = 252.$$

   (b) Let A, B be Alice and Bob, and let E, O be Eve and Oscar. There are 6 other people. We split into cases.

   Case 1: the committee contains both E and O. Then we choose 3 more people from the remaining 8 with the restriction that we cannot choose both A and B. There are $\binom{8}{3}$ ways to choose 3 more people from the remaining 8 to be on the committee; of these, there are $\binom{6}{1}$ ways in which both A and B are chosen (such a committee must have E, O, A, B so there is only one more person to choose from the remaining 6 people). Thus the number of committees for this case is

$$\binom{8}{3} - \binom{6}{1} = 56 - 6 = 50.$$

   Case 2: the committee contains neither E nor O. Then we choose 5 people from the remaining 8, again not allowing both A and B. There are $\binom{8}{5}$ ways to choose 5 people from the remaining 8 people to be on the committee; of these, there are $\binom{6}{3}$ ways in which both A and B are chosen. Thus the number of committees for this case is

$$\binom{8}{5} - \binom{6}{3} = 56 - 20 = 36.$$

   Hence the total number is $50 + 36 = 86$.

   (c) To have none of Alice, Bob, Eve, or Oscar on the committee, we must choose all 5 members from the other 6 people, which can be done in $\binom{6}{5} = 6$ ways. Therefore the probability is

$$\frac{6}{86} = 0.069767\ldots$$

2. (a) The positive two-digit multiples of 3 occur as every third number from 12 to 99, so there are

$$\frac{99 - 12}{3} + 1 = 30$$

such numbers.

(b) There are 90 positive two-digit integers (9 choices for the first digit and 10 choices for the second digit). Thus, by part (a), the probability that a randomly selected positive two-digit number is a multiple of 3 is

$$\frac{30}{90} = \frac{1}{3}.$$

(c) The positive two-digit multiples of 4 occur as every fourth number from 12 to 96, so there are

$$\frac{96-12}{4} + 1 = 22$$

such numbers. Hence the probability that a randomly selected positive two-digit number is a multiple of 4 is

$$\frac{22}{90} = \frac{11}{45}.$$

§40 — Binomial Coefficients

Epp 4th ed. pp. 565–591 · 5th ed. (metric) pp. 617–641. Lecture 31 (week 11).

Problems

1. What is the coefficient of x^4 in $(2+3x)^7$?
2. (a) What is the coefficient of x^{12} in $(3-x^3)^6$?
   (b) What is the coefficient of x^8 in $(3-x^3)^6$?
3. What is the coefficient of x^2 in the expansion of $(3-x)^6$?
4. What is the coefficient of x^5 in the expansion of $(2x+1)^{20}$?
5. What is the coefficient of x^5 in the expansion of $\left(3x^2 - \dfrac{2}{x}\right)^7$?

Solutions

1. By the binomial theorem,

$$(2+3x)^7 = \sum_{k=0}^{7} \binom{7}{k} 2^{7-k}(3x)^k.$$

The term x^4 appears only when $k=4$. This term is $\binom{7}{4}2^3(3x)^4$. Therefore the coefficient of x^4 is

$$\binom{7}{4}2^3 3^4 = 35 \times 8 \times 81 = 22\,680.$$

2. By the binomial theorem,

$$(3-x^3)^6 = \sum_{k=0}^{6} \binom{6}{k} 3^{6-k}(-x^3)^k = \sum_{k=0}^{6} \binom{6}{k} 3^{6-k}(-1)^k x^{3k}.$$

(a) The term involving x^{12} is $\binom{6}{4}3^2(-1)^4 x^{3\cdot 4}$, so the coefficient of x^{12} is

$$15 \times 9 \times (-1)^4 = 135.$$

(b) Every power of x that appears is a multiple of 3. Since 8 is not a multiple of 3, there is no x^8 term, so the coefficient of x^8 is $\mathbf{0}$.

3. By the binomial theorem,

$$(3-x)^6 = \sum_{i=0}^{6} \binom{6}{i} 3^{6-i}(-x)^i = \sum_{i=0}^{6} \binom{6}{i} 3^{6-i}(-1)^i x^i.$$

The term x^2 appears only when $i=2$, so the coefficient of x^2 is

$$\binom{6}{2}3^{6-2}(-1)^2 = \binom{6}{2}3^4 = 1215.$$

4. By the binomial theorem,

$$(2x+1)^{20} = \sum_{k=0}^{20} \binom{20}{k} (2x)^{20-k}(1)^k.$$

The term x^5 appears only when $k = 15$, where the term is $\binom{20}{15}(2x)^5$, so the coefficient of x^5 is

$$\binom{20}{5} 2^5 = 496\,128.$$

5. By the binomial theorem,

$$\left(3x^2 - \frac{2}{x}\right)^7 = \sum_{i=0}^{7} \binom{7}{i} (3x^2)^{7-i} \left(-\frac{2}{x}\right)^i = \sum_{i=0}^{7} \binom{7}{i} 3^{7-i}(-2)^i x^{14-3i}.$$

The term x^5 appears only when $14 - 3i = 5$, i.e. $i = 3$. Therefore the coefficient of x^5 is

$$\binom{7}{3} 3^4(-2)^3 = -22\,680.$$

§41 — Inclusion Exclusion

Epp 4th ed. pp. 545–549 · 5th ed. (metric) pp. 595–599. Lecture 32 (week 12).

Problems

1. Out of 100 people surveyed, 40 people like apples, 50 like bananas and 40 like cherries. 15 like apples and bananas, 16 like bananas and cherries, 17 like cherries and apples and 10 people like all three. How many people do not like any of the three fruits?
2. How many integers between 1 and 600 inclusive are coprime with 15? That is, how many integers n satisfy $1 \leq n \leq 600$ and $\gcd(n, 15) = 1$?
3. Seven friends, named Alice, Bob, Charlie, Delia, Edward, Fred and George, stand in a line for a photograph. How many different arrangements are there of the seven friends standing in a line such that Alice is not next to Charlie and Bob is not next to Delia?

Solutions

1. Let A be the set of people who like apples, B the set of people who like bananas, and C the set of people who like cherries. We are given

$$|A| = 40,\ |B| = 50,\ |C| = 40, \quad |A \cap B| = 15,\ |B \cap C| = 16,\ |C \cap A| = 17, \quad |A \cap B \cap C| = 10.$$

   By inclusion–exclusion,

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| \\ &= 40 + 50 + 40 - 15 - 17 - 16 + 10 \\ &= 92. \end{aligned}$$

   Hence the number of people who do not like any of the three fruits is

$$100 - |A \cup B \cup C| = 100 - 92 = 8.$$

2. Let $U = \{1, 2, \ldots, 600\}$. Define

$$A = \{n \in U : 3 \mid n\}, \qquad B = \{n \in U : 5 \mid n\}.$$

   Then n is not coprime to 15 if and only if $n \in A \cup B$. We have

$$|A| = \frac{600}{3} = 200, \qquad |B| = \frac{600}{5} = 120, \qquad |A \cap B| = \frac{600}{15} = 40.$$

   By inclusion–exclusion,

$$|A \cup B| = |A| + |B| - |A \cap B| = 200 + 120 - 40 = 280.$$

   Therefore the number of n with $\gcd(n, 15) = 1$ is

$$|U| - |A \cup B| = 600 - 280 = 320.$$

3. Let U be the set of all linear arrangements of the 7 friends, so $|U| = 7!$. Define

$$A = \{\text{arrangements in which Alice is next to Charlie}\},$$
$$B = \{\text{arrangements in which Bob is next to Delia}\}.$$

We want to count the number of arrangements in U which are neither in A nor in B. The required number is $|U \setminus (A \cup B)| = |U| - |A \cup B|$.

To count $|A|$, treat {Alice, Charlie} as a single block. There are 6! ways to arrange the 6 "objects", and for each of these there are 2 ways to order Alice and Charlie within their block (either Alice or Charlie to the left). Hence

$$|A| = 2 \cdot 6!.$$

Similarly, $|B| = 2 \cdot 6!$.

To count $|A \cap B|$, treat both pairs {Alice, Charlie} and {Bob, Delia} as two single blocks. There are 5! ways to arrange the 5 objects (two blocks plus the other three people), and for each of these there are $2 \cdot 2$ internal orders of the two blocks. Hence

$$|A \cap B| = 4 \cdot 5!.$$

By inclusion–exclusion,

$$|A \cup B| = |A| + |B| - |A \cap B| = 2 \cdot 6! + 2 \cdot 6! - 4 \cdot 5!.$$

Therefore

$$|U \setminus (A \cup B)| = 7! - (2 \cdot 6! + 2 \cdot 6! - 4 \cdot 5!) = 2640.$$

§42 — The Pigeonhole Principle

Epp 4th ed. pp. 554–565 · 5th ed. (metric) pp. 604–616. Lecture 33 (week 12).

Problems

1. (a) Explain why, in a group of 50 people, at least five people were born in the same month.

   (b) How many people do you need to ensure that at least seven people were born in the same month?

2. Show that, if we select six distinct integers from the set $\{1, 2, \ldots, 10\}$, two of these must add to give 11.

3. A computer network consists of six computers. Each computer is directly connected to zero or more of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

4. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

5. Let $S = \{1, 2, \ldots, 100\}$. What is the minimum number of integers you must pick from S to be sure that at least one pair has a difference which is divisible by 3?

Solutions

1. (a) There are 12 months (pigeonholes) and 50 people (pigeons). If each month had at most 4 birthdays, then there would be at most $12 \cdot 4 = 48$ people. Since $50 > 48$, by the pigeonhole principle some month must contain at least 5 people.

   (b) To guarantee at least 7 people are born in the same month, note that if each month had at most 6 birthdays, then there could be at most $12 \cdot 6 = 72$ people. Therefore with **73** people, by the pigeonhole principle some month has at least 7 people. Moreover, 72 people is not enough (take 6 born in each month), so the minimum is 73.

2. Partition $A = \{1, 2, \ldots, 10\}$ into the 5 pairs

$$\{1, 10\},\ \{2, 9\},\ \{3, 8\},\ \{4, 7\},\ \{5, 6\},$$

   each summing to 11. In order for a selection of distinct integers from the set A to **not** have two that add to give 11, at most one integer can be chosen from each of the subsets. Choosing 6 distinct integers from A forces us to select two numbers from the same pair (there are 5 pairs but 6 choices), by the pigeonhole principle. Those two numbers then add to 11.

3. For each computer, let its **degree** be the number of other computers it is directly connected to. With 6 computers, each degree is an integer in $\{0, 1, 2, 3, 4, 5\}$. It is impossible to have both a computer of degree 0 and a computer of degree 5 (if one is connected to everyone, then none has degree 0). Hence the degrees must all lie in one of the two 5-element sets

$$\{0, 1, 2, 3, 4\} \quad \text{or} \quad \{1, 2, 3, 4, 5\}.$$

So, in either case, there are only 5 possible degree values available for 6 computers. By the pigeonhole principle, at least two computers have the same degree, i.e. are connected to the same number of other computers.

4. Let g_i be the total number of games played in the first i days, for $i = 1, 2, \ldots, 30$. Then

$$1 \leq g_1 < g_2 < \cdots < g_{30} \leq 45$$

because at least one game is played each day. Consider also the 30 numbers

$$g_1 + 14,\ g_2 + 14,\ \ldots,\ g_{30} + 14,$$

which satisfy $15 \leq g_1 + 14 < \cdots < g_{30} + 14 \leq 59$. Thus we have 60 integers

$$g_1, \ldots, g_{30},\ g_1 + 14, \ldots, g_{30} + 14$$

all lying in the set $\{1, 2, \ldots, 59\}$ of size 59. By the pigeonhole principle, two of these 60 integers are equal. Since the g_i are strictly increasing, no equality can occur among the g_i themselves, and likewise no equality can occur among the $g_i + 14$ themselves. Therefore we must have

$$g_j = g_i + 14$$

for some $i, j \in \{1, \ldots, 30\}$. Then the number of games played from day $i + 1$ through day j is

$$g_j - g_i = 14,$$

so there is a period of consecutive days during which exactly 14 games are played.

5. First, we note that two integers have a difference that is divisible by 3 if and only if they are congruent modulo 3 (have the same remainder when divided by 3). Now, the set S can be partitioned into three sets based on their congruence class modulo 3. By the pigeonhole principle, if we select **4** numbers from S, at least two of them must belong to the same congruence class. These two will have difference that is divisible by 3. Note that 4 is the minimum number of selections that guarantees this, since it is possible to select 3 numbers where no two of them have difference divisible by 3 (select one from each congruence class).

§43 — Introduction to Graph Theory

Epp 4th ed. pp. 625–660 · 5th ed. (metric) pp. 677–697. Lecture 33 (week 12).

Problems

1. For each of the following, state whether or not there exists a simple graph with the stated number of vertices and degrees. Draw the graph, or explain why one does not exist.
   (a) Five vertices with degrees 3, 3, 2, 2, 2.
   (b) Seven vertices with degrees 4, 2, 3, 1, 1, 1, 1.
   (c) Five vertices with degrees 5, 4, 3, 1, 1.
2. A graph has 10 edges and 6 vertices. Five of the vertices have degree 3. What is the degree of the remaining vertex?
3. Let n be a positive integer and let K_n be the complete graph with n vertices. How many edges does K_n have?
4. A graph has 27 edges and every vertex has degree 2, 3 or 4. If there are 7 vertices that have degree 4 and 4 vertices that have degree 3, then how many vertices of degree 2 are there?

Solutions

1. (a) **Yes**, such a simple graph exists. One drawing has vertices a, b, c, d, e and the six edges

$$ab, \quad ac, \quad ad, \quad bc, \quad be, \quad de,$$

giving $\deg(a) = 3$, $\deg(b) = 3$, $\deg(c) = 2$, $\deg(d) = 2$, $\deg(e) = 2$ as required.

   (b) **No** such graph exists. If the degrees are 4, 2, 3, 1, 1, 1, 1, then the sum of degrees is

$$4 + 2 + 3 + 1 + 1 + 1 + 1 = 13,$$

which is odd. Remember (by the Handshake Theorem) that the sum of the degrees in any graph equals twice the number of edges, and so is always even. So such a graph cannot exist.

(c) **No** such graph exists. In a simple graph on 5 vertices, the largest possible degree is 4, but the sequence contains a vertex of degree 5, which is impossible. So there is no such simple graph.

2. Since the graph has 10 edges, the sum of all the degrees is $2 \cdot 10 = 20$. If five vertices have degree 3, the sum of their degrees is $5 \cdot 3 = 15$, so the remaining vertex has degree

$$20 - 15 = 5.$$

3. In K_n, an edge is determined by choosing 2 distinct vertices, so the number of edges is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

4. Let x be the number of vertices of degree 2. Since the graph has 27 edges, the sum of degrees is $2 \cdot 27 = 54$. Given that there are seven vertices of degree 4 and four vertices of degree 3, and all other vertices have degree 2, we have

$$2x + 4 \cdot 7 + 3 \cdot 4 = 54.$$

This means $x = 7$.

§44 — Walks, Trails and Circuits

Epp 4th ed. pp. 625–660 · 5th ed. (metric) pp. 677–697. Lecture 34 (week 12).

Problems

1. Let G and H be the graphs below.

   G has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and is drawn with a **loop at** v_2, a **loop at** v_5, and **two parallel edges between** v_1 **and** v_2. Its degree sequence is

   $$\deg(v_1) = 4, \quad \deg(v_2) = 6, \quad \deg(v_3) = 4, \quad \deg(v_4) = 2, \quad \deg(v_5) = 4,$$

   so G is connected with 10 edges.

   H has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and is a simple connected graph drawn as the 4-cycle $v_1v_2v_3v_4v_1$ with a single pendant edge v_3v_5, giving degrees

   $$\deg(v_1) = 2, \quad \deg(v_2) = 2, \quad \deg(v_3) = 3, \quad \deg(v_4) = 2, \quad \deg(v_5) = 1.$$

   For each of the following statements, explain why it is true or false.

   (a) G is a simple graph.

   (b) The sum of the degrees of the vertices of G is 18.

   (c) G contains an Euler circuit.

   (d) H contains an Euler trail.

2. Does the following graph contain an Euler circuit? (The graph shown is connected and has exactly two vertices of degree 3.)

3. Let G be a simple, connected, Eulerian graph with eight vertices and let e be the number of edges in G. Prove that $8 \leq e \leq 24$.

Solutions

1. (a) This statement is **false**. The graph G has loops (at v_2 and v_5) and it also has multiple edges between v_1 and v_2, so G is not simple.

   (b) This statement is **false**. A loop contributes 2 to the degree, so we have

   $$\deg(v_1) = 4, \quad \deg(v_2) = 6, \quad \deg(v_3) = 4, \quad \deg(v_4) = 2, \quad \deg(v_5) = 4,$$

   and thus the sum of the degrees is $4 + 6 + 4 + 2 + 4 = 20 \neq 18$.

   (c) This statement is **true**. The graph G is connected and every vertex has even degree (as computed above), so G contains an Euler circuit.

(d) This statement is **true**. In H, the degrees are

$$\deg(v_1) = 2,\ \deg(v_2) = 2,\ \deg(v_4) = 2,\ \deg(v_3) = 3,\ \deg(v_5) = 1,$$

so exactly two vertices have odd degree (namely v_3 and v_5). Therefore H has an Euler trail (starting at one of v_3, v_5 and ending at the other).

2. **No.** The graph is connected, but has two vertices of degree 3, so not all vertices have even degree. Hence there is no Euler circuit.

3. Since G is connected and Eulerian, every vertex has even degree, and (because G is simple and connected with 8 vertices) each vertex has degree at least 2. Hence

$$2e = \sum_{v} \deg(v) \geq 8 \cdot 2 = 16,$$

so $e \geq 8$. Moreover, in a simple graph on 8 vertices we have $\deg(v) \leq 7$, and because G is Eulerian all degrees are even, so in fact $\deg(v) \leq 6$ for every vertex. Therefore

$$2e = \sum_{v} \deg(v) \leq 8 \cdot 6 = 48,$$

so $e \leq 24$. Hence $8 \leq e \leq 24$. □

§45 — Matrix Representations of Graphs

Epp 4th ed. pp. 661–675 · 5th ed. (metric) pp. 698–712. Lecture 35 (week 13).

Problems

1. Let G be the graph with vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{a, b, c, d, e, f\}$, where the edges are

edge	endpoints
a	$v_1 v_2$
b	$v_2 v_3$
c	$v_2 v_3$ (parallel to b)
d	$v_3 v_4$
e	$v_2 v_4$
f	loop at v_4

(a) Write down an adjacency matrix for G. Make sure you clearly label the rows and columns of your matrix.

(b) Write down an incidence matrix for G. Make sure you clearly label the rows and columns of your matrix.

(c) Does G have an Euler circuit? If yes, write it out as a sequence of vertices and edges. If not, determine the fewest number of edges that would need to be added to G to create a new graph that does have an Euler circuit.

2. Let G be the graph with the following adjacency matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(a) Draw the graph G.

(b) Explain whether or not G has an Euler circuit or an Euler trail.

3. Let G be the graph with the following adjacency matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Explain whether or not G is Eulerian.

Solutions

1. (a) The labelled adjacency matrix for G is

	v_1	v_2	v_3	v_4
$\mathbf{v_1}$	0	1	0	0
$\mathbf{v_2}$	1	0	2	1
$\mathbf{v_3}$	0	2	0	1
$\mathbf{v_4}$	0	1	1	1

(where the entry 1 on the diagonal at v_4 records the single loop at v_4).

(b) The labelled incidence matrix for G is

	a	b	c	d	e	f
$\mathbf{v_1}$	1	0	0	0	0	0
$\mathbf{v_2}$	1	1	1	0	1	0
$\mathbf{v_3}$	0	1	1	1	0	0
$\mathbf{v_4}$	0	0	0	1	1	2

(where the entry 2 in the bottom right indicates the loop that is incident twice with v_4).

(c) The degrees are

$$\deg(v_1) = 1, \quad \deg(v_2) = 4, \quad \deg(v_3) = 3, \quad \deg(v_4) = 4,$$

so G does **not** have an Euler circuit (since v_1 and v_3 have odd degree).

The fewest edges to add to make all degrees even is **1**; for example, adding an edge g between vertices v_1 and v_3 creates a new connected graph in which all of the vertices have even degree.

An example of an Euler circuit in the new graph is

$$v_1,\ a,\ v_2,\ b,\ v_3,\ d,\ v_4,\ f,\ v_4,\ e,\ v_2,\ c,\ v_3,\ g,\ v_1.$$

2. (a) Reading the matrix off (with vertices $v_1, \ldots, v_5$ in matrix order), G has the edges

- v_1v_2 — a single edge,
- v_1v_5 — **three** parallel edges,
- v_2v_3 — **two** parallel edges,
- v_3v_4 — a single edge,
- v_3v_5 — a single edge,
- v_4v_5 — a single edge.

(b) The degrees are

$$\deg(v_1) = 4,\ \deg(v_2) = 3,\ \deg(v_3) = 4,\ \deg(v_4) = 2,\ \deg(v_5) = 5,$$

so there is **no Euler circuit** (since v_2 and v_5 have odd degree), but there **is an Euler trail** (exactly two odd-degree vertices) from v_2 to v_5.

3. Let the vertices be $v_1, v_2, v_3, v_4, v_5, v_6$ in the order of the matrix. Reading the matrix off, G has edges

$$v_1v_2, \quad v_1v_4, \quad v_2v_4, \quad v_3v_5, \quad v_3v_6, \quad v_5v_6,$$

that is, two disjoint triangles: one on $\{v_1, v_2, v_4\}$ and one on $\{v_3, v_5, v_6\}$.

The graph is **not connected** (e.g. there is no walk from vertex v_1 to vertex v_3). Hence G is not Eulerian.

(Errata note: the source document's solution to 1(b) was corrected on 11 August 2026 — the note in brackets had previously copied the note from 1(a). The corrected note is the one given above.)

§46 — Trees

Epp 4th ed. pp. 683–701 · 5th ed. (metric) pp. 720–742. Lecture 36 (week 13).

Problems

1. A tree has 11 vertices, of which exactly four vertices have degree 1 and exactly one vertex has degree 4. Determine the degrees of the remaining six vertices.
2. Let T be a tree with p vertices, where $p \geq 11$. Suppose T has precisely six vertices of degree 1 and precisely four vertices of degree 3. Determine the degree of each of the remaining $p - 10$ vertices of T.
3. Let T be a tree on 12 vertices. Suppose T has exactly three vertices of degree 3 and each of the remaining vertices has degree 1 or 5. Using theorems that were given in lectures, determine the number of vertices of degree 5.
4. Two trees with the same vertex set $\{1, 2, \ldots, n\}$ are **distinct** if their edge sets are not equal. Use mathematical induction to prove that for each integer $n \geq 2$, the number of distinct trees with vertex set $\{1, 2, \ldots, n\}$ is at least $(n-1)!$.

Solutions

1. A tree with 11 vertices has $e = 10$ edges, so by the Handshake Theorem, the sum of degrees is $2e = 20$. We are given that exactly four vertices have degree 1 and exactly one vertex has degree 4. Let $v_1, v_2, \ldots, v_6$ be the remaining vertices whose degrees are to be determined. We have

$$\sum_{i=1}^{6} \deg(v_i) + 4 \cdot 1 + 1 \cdot 4 = 20$$

$$\sum_{i=1}^{6} \deg(v_i) = 12.$$

None of these six vertices $v_1, v_2, \ldots, v_6$ can have degree 1 (since there are exactly four vertices of degree 1, already accounted for). Hence each has degree at least 2. Because six vertices of degree at least 2 have total degree 12, each must have degree exactly 2.

Thus the remaining six vertices all have **degree 2**.

2. T is a tree with p vertices and so must have $p-1$ edges. Now by the Handshake Theorem we know

$$2(p-1) = \sum_{i=1}^{p} \deg(v_i).$$

Using the known degrees we can write this as

$$\begin{aligned} 2(p-1) &= 6 \cdot 1 + 4 \cdot 3 + \sum_{i=1}^{p-10} \deg(v_i) \\ 2(p-1) &= 18 + \sum_{i=1}^{p-10} \deg(v_i) \\ 2p - 2 - 18 &= \sum_{i=1}^{p-10} \deg(v_i) \\ 2(p-10) &= \sum_{i=1}^{p-10} \deg(v_i). \end{aligned}$$

Rewriting the final line as

$$\frac{\sum_{i=1}^{p-10} \deg(v_i)}{p-10} = 2, \qquad \text{i.e.} \qquad \frac{\deg(v_1) + \deg(v_2) + \cdots + \deg(v_{p-10})}{p-10} = 2,$$

we see that the average degree of the remaining $p-10$ vertices is 2. However, we know that there are no more vertices of degree 1. Thus each of the remaining $p-10$ vertices must have **degree exactly 2**.

3. Let x be the number of vertices of degree 5 and y the number of vertices of degree 1. Then, because the tree has 12 vertices, we have

$$x + y + 3 = 12.$$

This simplifies to give $x + y = 9$.

Now, since T is a tree on 12 vertices, it must have $e = 11$ edges. By the Handshake Theorem, the sum of degrees is $2e = 22$, so

$$3 \cdot 3 + 5x + 1 \cdot y = 22.$$

This simplifies to give $5x + y = 13$.

Subtracting $x + y = 9$ from $5x + y = 13$ gives $4x = 4$, hence $x = 1$. Therefore there is exactly **one** vertex of degree 5.

4. We prove by induction that the number of distinct trees with vertex set $\{1, 2, \dots, n\}$ is at least $(n-1)!$.

Basis: let $n = 2$. Here $(n-1)! = (2-1)! = 1$. There is exactly one tree on $\{1, 2\}$, namely the single edge $\{1, 2\}$. So the base case is true.

Inductive hypothesis. Suppose $k \geq 2$ is an integer and there are at least $(k-1)!$ distinct trees on $\{1, 2, \dots, k\}$.

Now consider $n = k + 1$. We aim to show that there are at least $k!$ distinct trees on vertex set $\{1, 2, \dots, k, k+1\}$.

Let T be a tree on vertex set $\{1, 2, \dots, k\}$. For any choice of $m \in \{1, 2, \dots, k\}$, we form a graph T_m on vertex set $\{1, 2, \dots, k, k+1\}$ as follows: start with tree T, add a new vertex $k+1$, and the edge $\{k+1, m\}$.

Note that T_m is connected (since $k+1$ is joined to m, and T was connected). Also, T_m has no circuit because T is a tree and adding a new leaf does not create a circuit. Thus T_m is a tree. Moreover, different choices of m give rise to distinct trees T_m, because their edge sets are different.

By the inductive hypothesis, there are at least $(k-1)!$ choices for the tree T. Since there are k choices for the vertex m, there are at least $k \cdot (k-1)! = k!$ trees T_m on vertex set $\{1, 2, \dots, k+1\}$.

By the principle of mathematical induction, this completes the proof. $\square$

See also

- math1061[110] — assessment weights and the full lecture schedule
- Logic: logical-connectives[111] · logical-equivalence-laws[112] · conditional-statements[113] · valid-argument-forms[114] · determining-argument-validity[115] · quantified-statements[116]
- Number theory and proof: proof-techniques[117] · rational-and-irrational-numbers[118] · divisibility-and-factorisation[119] · modular-arithmetic[120] · gcd-lcm-and-euclidean-algorithm[121]
- Graph theory (§43–§46): graphs[122] · walks-trails-and-circuits[123] · euler-circuits-and-trails[124] · adjacency-matrices[125] · incidence-matrices[126] · trees[127]

Proof Techniques — Direct, Counterexample, Contradiction, Contraposition

The four methods for establishing (or demolishing) a statement of the form $\forall x \in D, P(x) \to Q(x)$, plus the definitions every proof in this course is expected to unfold.

Definitions to prove from

Proofs are written by **demonstrating facts using the definitions** — so these have to be at your fingertips. Each is a biconditional: it can be used in either direction.

Term	Definition
n is **even**	$n = 2k$ for some integer k
n is **odd**	$n = 2k + 1$ for some integer k (equivalently $n = 2k - 1$ for some integer k)

[110] courses/math1061/math1061.pdf
[111] courses/math1061/logical-connectives.pdf
[112] courses/math1061/logical-equivalence-laws.pdf
[113] courses/math1061/conditional-statements.pdf
[114] courses/math1061/valid-argument-forms.pdf
[115] courses/math1061/determining-argument-validity.pdf
[116] courses/math1061/quantified-statements.pdf
[117] courses/math1061/proof-techniques.pdf
[118] courses/math1061/rational-and-irrational-numbers.pdf
[119] courses/math1061/divisibility-and-factorisation.pdf
[120] courses/math1061/modular-arithmetic.pdf
[121] courses/math1061/gcd-lcm-and-euclidean-algorithm.pdf
[122] courses/math1061/graphs.pdf
[123] courses/math1061/walks-trails-and-circuits.pdf
[124] courses/math1061/euler-circuits-and-trails.pdf
[125] courses/math1061/adjacency-matrices.pdf
[126] courses/math1061/incidence-matrices.pdf
[127] courses/math1061/trees.pdf

Term	Definition
n is **prime**	$n \in \mathbb{Z}$, $n > 1$, and $\forall r, s \in \mathbb{Z}^+$, if $n = rs$ then $(r = 1$ and $s = n)$ or $(r = n$ and $s = 1)$
n is **composite**	$n \in \mathbb{Z}$, $n > 1$, and $\exists r, s \in \mathbb{Z}^+$ such that $n = rs$ and $1 < r < n$ and $1 < s < n$
r is **rational**	$\exists a, b \in \mathbb{Z}$ such that $r = \frac{a}{b}$ and $b \neq 0$
$d \mid n$	$d \neq 0$ and $n = dk$ for some integer k

Two facts used constantly, and worth naming explicitly when you lean on them:

- **Fact**: the sum, difference, and product of integers is an integer.
- **Fact**: every integer is either even or odd.

Parity

Almost every early proof in this course turns on whether something is even or odd, so the consequences of those two definitions are worth having ready rather than re-deriving:

	Result
even + even, odd + odd	even
even + odd	odd
even× anything	even
odd × odd	odd
consecutive integers	opposite parity — so two integers **two apart share their parity**

Each row follows from the two definitions by a one-line substitution. Subtraction behaves like addition throughout, since $m - n$ and $m + n$ differ by $2n$.

The "two apart" row is the whole content of the warm-up in 2026-07-27-welcome-and-intro-problems[128], where consecutive $p < q < r$ forces $p + r$ to be even and $r - p$ to be exactly 2, while pr is free to be either.

Prime vs. not-prime. For $n \in \mathbb{Z}^{>1}$, "n is not prime" means "n is composite". For $n \in \mathbb{Z}$ generally, "n is not prime" means "n is composite **or** $n \leq 1$" — the integers split into $\mathbb{Z}^{\leq 1}$, $\mathbb{Z}^{\text{prime}}$, and $\mathbb{Z}^{\text{composite}}$, and 1 and everything below it is in none of the latter two. This is why $n = 1$ is the standard counterexample to "every positive integer is prime or composite".

Warning: don't assume something that seems like an "obvious fact" until you have proven it, or can name the lecture or pre-work video it was proved in.

[128] courses/math1061/2026-07-27-welcome-and-intro-problems.pdf

The four methods

All four target $\forall x \in D, P(x) \to Q(x)$. Recall the truth table for $p \to q$ — the only row that makes it false is p true, q false:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Every method below is a different way of attacking that one row.

Direct proof (shows the statement is true)

> Let $x \in D$ and suppose $P(x)$. ... Hence $Q(x)$. $\square$

Why suppose $P(x)$ is true, when $P(x) \to Q(x)$ holds anyway if $P(x)$ is false? Because the only case that could make the statement false is $P(x)$ true and $Q(x)$ false. Ruling that out is the whole job — the $P(x)$-false rows are true for free.

Disproof by counterexample (shows the statement is false)

> Let $x = \cdots$. Then $x \in D$ and $P(x)$ is true. However, $Q(x)$ is false. So x is a counterexample and the statement is false.

A counterexample to $p \to q$ needs p **true** and q **false** — a value where the hypothesis fails proves nothing.

Proof by contradiction (shows the statement is true)

> Suppose the negation of $\forall x \in D, P(x) \to Q(x)$ is true. That is, suppose $\exists x \in D$ such that $P(x)$ is true but $Q(x)$ is false. ... Hence something that is obviously wrong, like $P(x)$ is false, or $Q(x)$ is true, or $1 = 0$. This is a contradiction, and we conclude that $\forall x \in D, P(x) \to Q(x)$ is true. $\square$

The set-up comes straight from quantified-statements[129]: the negation of $\forall x \in D, P(x) \to Q(x)$ is $\exists x \in D$ such that $P(x) \wedge \sim Q(x)$.

[129] courses/math1061/quantified-statements.pdf

Proofs of the statement $\forall x \in D,\ P(x) \to Q(x)$

Direct proof:

Let $x \in D$ and suppose $P(x)$.

...

Hence $Q(x)$. □

Shows the statement is true.

Disproof by counterexample:

Let $x = \cdots$. Then $x \in D$ and $P(x)$ is true.

However, $Q(x)$ is false.

So x is a counterexample and the statement is false.

Shows the statement is false.

Proof by contradiction:

Suppose the negation of $\forall x \in D, P(x) \to Q(x)$ is true.

That is, suppose $\exists x \in D$ such that $P(x)$ is true but $Q(x)$ is false.

...

Hence something that is obviously wrong, like $P(x)$ is false, or $Q(x)$ is true, or $1 = 0$.

This is a contradiction, and we conclude that $\forall x \in D, P(x) \to Q(x)$ is true. □

Shows the statement is true.

Proof by contraposition:

We prove the contrapositive of $\forall x \in D, P(x) \to Q(x)$, namely that $\forall x \in D, \sim Q(x) \to \sim P(x)$.

Let $x \in D$ and suppose $\sim Q(x)$.

...

Hence, $\sim P(x)$. □

Figure 12: The skeletons of the four methods for a statement of the form x D, P(x) → Q(x). Three of the four establish the statement and one demolishes it, which is what the red verdict lines record.

Useful negations to have ready (see logical-equivalence-laws[130]):

$$\sim (p \vee q) \equiv \sim p \wedge \sim q \qquad \sim (p \wedge q) \equiv \sim p \vee \sim q \qquad \sim (p \to q) \equiv p \wedge \sim q$$

Proof by contraposition (shows the statement is true)

> We prove the contrapositive of $\forall x \in D, P(x) \to Q(x)$, namely that $\forall x \in D, \sim Q(x) \to \sim P(x)$. Let $x \in D$ and suppose $\sim Q(x)$. ... Hence, $\sim P(x)$. $\square$

This is valid because $p \to q \equiv \sim q \to \sim p$ (see conditional-statements[131]).

Choosing a method

- **Contradiction is the most versatile**, but it needs more set-up, so a direct proof is often easier when one is available.
- **Contraposition** is the natural choice when $\sim Q(x)$ gives you something concrete to work with and $P(x)$ doesn't — typically when the conclusion is a negative statement ("r is irrational", "n is odd") whose negation is a definition you can expand.
- **Contradiction vs. contraposition**: a contraposition proof is a contradiction proof that never actually needs the assumption $P(x)$ — you derive $\sim P(x)$ from $\sim Q(x)$ alone. If your "contradiction" proof only ever uses the $\sim Q(x)$ half of the assumption, write it as a contraposition instead.
- **Counterexample vs. contradiction**: counterexample is a **disproof**, contradiction is a **proof**. They are not variants of each other.

Worked comparisons on the same statements live in the lecture notes: 2026-08-13-proof-by-contradiction[132] proves $\forall n \in \mathbb{Z}$, if n is odd then $3n+2$ is odd both directly and by contradiction, and 2026-08-13-proof-by-contraposition[133] works through statements where one method gets stuck and another doesn't.

Worked examples

Direct proof

$\forall n \in \mathbb{Z}$, if n is odd then $3n + 2$ is odd.

[130] courses/math1061/logical-equivalence-laws.pdf
[131] courses/math1061/conditional-statements.pdf
[132] courses/math1061/2026-08-13-proof-by-contradiction.pdf
[133] courses/math1061/2026-08-13-proof-by-contraposition.pdf

Let $n \in \mathbb{Z}$ and suppose n is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. Now

$$3n + 2 = 3(2k+1) + 2 = 6k + 5 = 2(3k+2) + 1.$$

Since $k \in \mathbb{Z}$, we have $3k + 2 \in \mathbb{Z}$, and hence $3n + 2$ is odd. □

For any integer x, if $x + 6 = 4y$ for some integer y, then $\frac{x}{2}$ is an odd integer.

Let $x \in \mathbb{Z}$ and suppose $x + 6 = 4y$ for some $y \in \mathbb{Z}$. Then $x = 4y - 6 = 2(2y - 3)$, so

$$\frac{x}{2} = \frac{2(2y-3)}{2} = 2y - 3 = 2(y-1) - 1.$$

Since $y \in \mathbb{Z}$ we have $2y - 3 \in \mathbb{Z}$, so $\frac{x}{2} \in \mathbb{Z}$; and $y - 1 \in \mathbb{Z}$, so by definition $\frac{x}{2}$ is odd. □

The sum of any pair of even integers is even.

Let $m, n \in \mathbb{Z}$ be even. Then $m = 2k$ and $n = 2\ell$ for some $k, \ell \in \mathbb{Z}$. Hence $m + n = 2k + 2\ell = 2(k + \ell)$, which is even. □

$\forall n \in \mathbb{Z}^+$, if $n \geq 4$ then $2n^2 - 5n + 2$ is composite.

Suppose $n \in \mathbb{Z}^+$ and $n \geq 4$. Factor $2n^2 - 5n + 2 = (2n-1)(n-2)$. Since $n \geq 4$, we have $2n - 1 \geq 7$ and $n - 2 \geq 2$, so both factors are integers greater than 1. Hence the expression is composite. □

Disproof by counterexample

- **For all integers m and n, if $2m + n$ is odd then m and n are both odd.** Counterexample $m = 4, n = 5$: $2(4) + 5 = 13$ is odd, but m is even. (Not $m = 3, n = 5$ — there the conclusion holds; not $m = 3, n = 6$ or $m = 4, n = 6$ — there the hypothesis fails.)
- **$\forall n \in \mathbb{Z}^{\geq 2}$, n is composite or $n+1$ is composite.** Take $n = 2$: both 2 and 3 are prime.
- **For each integer $n \geq 2$, the product of the first n primes minus 1 is prime.** Take $n = 4$: $2 \cdot 3 \cdot 5 \cdot 7 - 1 = 209 = 11 \cdot 19$, composite.
- **$\forall x, y \in \mathbb{R}$, if $y^2 > x^2$ then $y > x$.** Take $x = 1, y = -2$: $4 > 1$ but $-2 \not> 1$.
- **$\forall x \in \mathbb{R}, \lfloor x^2 \rfloor = \lfloor x \rfloor^2$.** Take $x = \frac{3}{2}$: $\lfloor \frac{9}{4} \rfloor = 2$ but $\lfloor \frac{3}{2} \rfloor^2 = 1$. (See modular-arithmetic[134] for the floor function.)

[134] courses/math1061/modular-arithmetic.pdf

Disproof by proving the negation

When the statement to disprove is existential, a single counterexample isn't available — you must prove the universal negation.

$\exists n \in \mathbb{Z}^+$ **such that** $n^2 + 5n + 6$ **is prime.** This is false; the negation is $\forall n \in \mathbb{Z}^+$, $n^2 + 5n + 6$ is composite.

> Let $n \in \mathbb{Z}^+$. Now $n^2 + 5n + 6 = (n + 2)(n + 3)$. Since $n \geq 1$ we have $n + 2 > 1$ and $n + 3 > 1$, so $n^2 + 5n + 6$ is composite. $\square$

(Because $n \geq 1$ forces $n^2 + 5n + 6 \geq 2$, "not prime" here does mean "composite".)

There is an even integer n **such that** $5n - 4$ **is prime.** False; prove $\forall n \in \mathbb{Z}^{\text{even}}$, $5n - 4$ is not prime.

> Suppose n is even, so $n = 2k$ for some $k \in \mathbb{Z}$. Then $5n - 4 = 10k - 4 = 2(5k - 2)$, which is even. The only even prime is 2; if $2(5k-2) = 2$ then $k = \frac{3}{5} \notin \mathbb{Z}$. Therefore $5n - 4$ is not prime. $\square$

Proof by contradiction

$\forall m, n \in \mathbb{Z}$**, if** mn **is even, then** m **is even or** n **is even.**

> Suppose $\exists m, n \in \mathbb{Z}$ such that mn is even and both m and n are odd. Since m is odd, $m = 2k + 1$ for some $k \in \mathbb{Z}$; since n is odd, $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$ — **a new variable**, since m and n need not be equal. Now
>
> $$mn = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1.$$
>
> Since $k, \ell \in \mathbb{Z}$, $2k\ell + k + \ell \in \mathbb{Z}$, so mn is odd. It is impossible for mn to be both even and odd, a contradiction. Therefore the original statement is true. $\square$

$\forall n \in \mathbb{Z}$**, if** $3n^3 - 2$ **is odd, then** n **is odd.**

> Suppose the statement is false: $\exists n \in \mathbb{Z}$ such that $3n^3 - 2$ is odd and n is even. Since n is even, $n = 2a$ for some $a \in \mathbb{Z}$. Then
>
> $$3n^3 - 2 = 3(2a)^3 - 2 = 24a^3 - 2 = 2(12a^3 - 1).$$
>
> Since $a \in \mathbb{Z}$, $12a^3 - 1 \in \mathbb{Z}$, so $3n^3 - 2$ is even — a contradiction. Therefore the original statement is true. $\square$

$\forall m, n \in \mathbb{Z}$**, if** $m + n$ **is even, then** m **and** n **are both even or both odd.**

Suppose the statement is false. Then $\exists m, n \in \mathbb{Z}$ such that $m+n$ is even but exactly one of m, n is even. **Without loss of generality**, assume m is even and n is odd — the other case (m odd, n even) follows because $m+n = n+m$. Thus $m = 2a$ for some $a \in \mathbb{Z}$ and $n = 2b+1$ for some $b \in \mathbb{Z}$. Now $m+n = 2a+2b+1 = 2(a+b)+1$, and $a+b \in \mathbb{Z}$, so $m+n$ is odd. This contradicts the assumption that $m+n$ is even. Therefore the original statement is true. □

For integers a and b, if $6a+3b$ is odd, then b is odd.

Suppose there exist integers a, b with $6a+3b$ odd and b even. Then $b = 2k$ for some $k \in \mathbb{Z}$, so $6a+3b = 6a+6k = 2(3a+3k)$, which is even — contradicting that $6a+3b$ is odd. □

For all positive integers x and y, $x^2 - y^2 \neq 1$.

Suppose $x, y \in \mathbb{Z}^+$ with $x^2 - y^2 = 1$. Factoring, $1 = (x-y)(x+y)$. Since x, y are positive integers we must have $x-y = 1$ and $x+y = 1$, so $2x = 2$ and $x = 1$; then $y = 0$, which is not a positive integer — a contradiction. □

Proof by contraposition

$\forall r \in \mathbb{R}$, if r^2 is irrational, then r is irrational.

We prove the contrapositive: $\forall r \in \mathbb{R}$, if $r \in \mathbb{Q}$ then $r^2 \in \mathbb{Q}$. Suppose $r \in \mathbb{R}$ with $r \in \mathbb{Q}$. Then $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Now $r^2 = \frac{a^2}{b^2}$. Since $a, b \in \mathbb{Z}$ we have $a^2, b^2 \in \mathbb{Z}$, and $b^2 \neq 0$ since $b \neq 0$. So $r^2 \in \mathbb{Q}$. □

$\forall m, n \in \mathbb{Z}$, if mn is odd, then m and n are both odd.

We prove the contrapositive: $\forall m, n \in \mathbb{Z}$, if at least one of m, n is even then mn is even. Without loss of generality, assume m is even, so $m = 2k$ for some $k \in \mathbb{Z}$. Now $mn = (2k)n = 2(kn)$, and $kn \in \mathbb{Z}$, so mn is even. □

For all integers a and b, if $(ab)^2$ is odd then a is odd and b is odd.

Contrapositive: if a is even or b is even, then $(ab)^2$ is even. WLOG suppose a is even, so $a = 2k$ for some $k \in \mathbb{Z}$. Then $ab = 2kb$ and $(ab)^2 = 4k^2b^2 = 2(2k^2b^2)$. Since $k, b \in \mathbb{Z}$, $2k^2b^2 \in \mathbb{Z}$, so $(ab)^2$ is even. □

For all integers x and y, if $x^2(y^2-2y)$ is odd then x and y are odd.

Contrapositive: if x is even or y is even, then $x^2(y^2-2y)$ is even. Here the two cases need separate treatment.

Case 1: x even. $x = 2k$ for some $k \in \mathbb{Z}$, so $x^2(y^2-2y) = 4k^2(y^2-2y) = 2(2k^2(y^2-2y))$, and $2k^2(y^2-2y) \in \mathbb{Z}$.

Case 2: y even. $y = 2\ell$ for some $\ell \in \mathbb{Z}$, so $x^2(y^2 - 2y) = x^2(4\ell^2 - 4\ell) = 2(x^2(2\ell^2 - 2\ell))$, and $x^2(2\ell^2 - 2\ell) \in \mathbb{Z}$.

In both cases $x^2(y^2 - 2y)$ is even. □

For any integer n, n^2 is odd if and only if n is odd. A biconditional needs both directions, and they want different methods:

($\Leftarrow$, direct) Suppose n is odd, so $n = 2k+1$. Then $n^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$ is odd.

($\Rightarrow$, contraposition) The contrapositive is: if n is even then n^2 is even. Suppose $n = 2k$. Then $n^2 = 4k^2 = 2(2k^2)$ is even. □

This lemma — **for all integers n, if n^2 is even then n is even** — is used in the proof that $\sqrt{2}$ is irrational (see rational-and-irrational-numbers[135]).

Common proof errors

The course explicitly tests spotting these. The seven below are the list given with the practice problems' §9 Q3 (see practice-problems[136], where the four proofs you are asked to find the errors in are worked):

- **Arguing from examples** — a single example never establishes a universal statement. ("$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy = 1$. Proof: if $x = 10$ take $y = \frac{1}{10}$." Not a proof — and the statement is false, since $x = 0$ has no such y.)
- **Using the same variable to mean two different things** — "let $m = 2k + 1$ and $n = 2k$" only proves the case $m = n + 1$. Use a fresh variable for each quantity.
- **Jumping to the conclusion** — writing down the definition you need to satisfy and then asserting it holds, without exhibiting the witnesses.
- **Assuming what is to be proved** — starting from "$4a + 2b = 2\ell$" and manipulating both sides assumes the conclusion. Derive the conclusion; don't begin with it.
- **Confusion between what is known and what is to be shown.**
- **Use of the word *any* instead of *some*.**
- **Misuse of the word *if*.**

One more, not on that list: **forgetting to state your assumptions**, or forgetting to say where a variable lives ("for some $r, s \in \mathbb{Z}$"). Introduce every existential witness with its domain.

[135] courses/math1061/rational-and-irrational-numbers.pdf
[136] courses/math1061/practice-problems.pdf

See also

- quantified-statements[137] — the negations these proofs are built on
- conditional-statements[138] · logical-equivalence-laws[139]
- 2026-07-27-welcome-and-intro-problems[140] — the parity warm-up, the course's first proof-shaped problem
- proving-set-properties[141] — the element method, and these techniques applied to sets
- rational-and-irrational-numbers[142] · divisibility-and-factorisation[143]
- practice-problems[144] — §9 Direct Proofs and Counterexamples, with full worked solutions
- practice-problems[145] — §10 Proof by Contradiction, with full worked solutions
- practice-problems[146] — §11 Proof by Contraposition, with full worked solutions

Quantified Statements

Where logical-connectives[147] and conditional-statements[148] deal with whole statements, **predicates** let us talk about statements whose truth depends on a variable, and **quantifiers** turn those predicates back into statements that are definitely true or false.

Predicates

A **predicate** is a sentence with a finite number of variables that becomes a statement when particular values are assigned to those variables.

- Predicates: $x > 0$, "x divides y".
- Not predicates: "Is it raining?" (a question), "She is happy." (no objective truth value).

A predicate is written $P(x)$, and the set of values x may take is its **domain** D.

Common domains

[137] courses/math1061/quantified-statements.pdf
[138] courses/math1061/conditional-statements.pdf
[139] courses/math1061/logical-equivalence-laws.pdf
[140] courses/math1061/2026-07-27-welcome-and-intro-problems.pdf
[141] courses/math1061/proving-set-properties.pdf
[142] courses/math1061/rational-and-irrational-numbers.pdf
[143] courses/math1061/divisibility-and-factorisation.pdf
[144] courses/math1061/practice-problems.pdf
[145] courses/math1061/practice-problems.pdf
[146] courses/math1061/practice-problems.pdf
[147] courses/math1061/logical-connectives.pdf
[148] courses/math1061/conditional-statements.pdf

Symbol	Set
$\mathbb{R}$	real numbers
$\mathbb{Q}$	rational numbers
$\overline{\mathbb{Q}}$	irrational numbers ($\mathbb{R} \setminus \mathbb{Q}$)
$\mathbb{Z}$	integers, $\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{N}$	natural numbers
$\mathbb{Z}^+$	positive integers, $\{1, 2, 3, \dots\}$
$\mathbb{Z}^{\text{nonneg}} = \mathbb{Z}^{\geq 0}$	$\{0, 1, 2, 3, \dots\}$
$\mathbb{Z}^{\text{even}}$	$\{\dots, -4, -2, 0, 2, 4, \dots\}$

Nesting: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, with $\overline{\mathbb{Q}}$ sitting alongside $\mathbb{Q}$ inside $\mathbb{R}$.

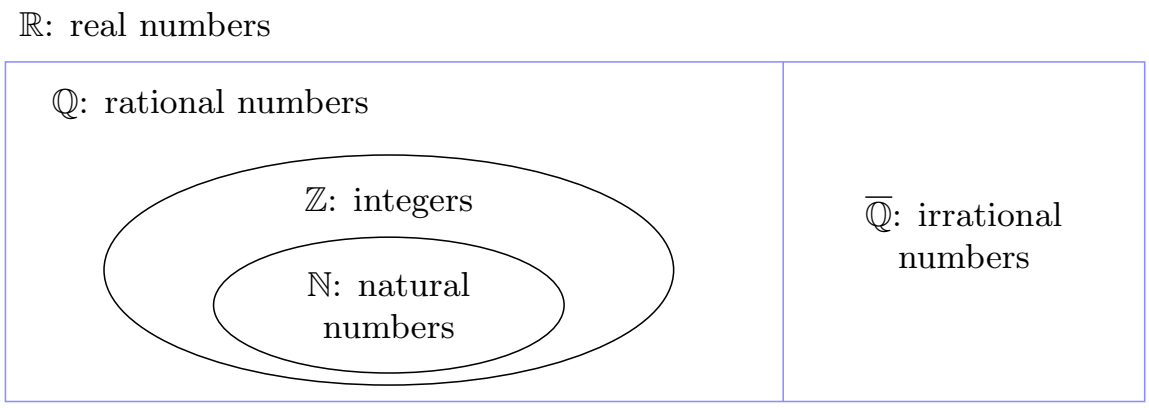


Figure 13: The domains as containers. Each nested label sits inside its own region, with $\mathbb{R}$ named above the frame, so a claim like 'every integer is rational' is read off as one curve lying inside another. The two panels show that the rationals and the irrationals divide the reals between them with nothing in both and nothing left over, which is why 'x is not rational' and 'x is irrational' say the same thing about a real number — but only about a real number.

Take care with $\mathbb{N}$ — the diagram deliberately does not say where 0 goes. Different sources define $\mathbb{N}$ as $\{0, 1, 2, \dots\}$ *or* $\{1, 2, 3, \dots\}$, and the course does not pick one. Prefer the unambiguous $\mathbb{Z}^+$ or $\mathbb{Z}^{\geq 0}$ in your own writing.

$\in$ means "belongs to" / "is an element of"; $\notin$ means "is not an element of". Note that $\{\mathbb{Z}\} \neq \mathbb{Z}$ — writing $x \in \{\mathbb{Z}\}$ says x *is* the set $\mathbb{Z}$, not that x is an integer.

The two quantifiers

Symbol	Reads	Associated English words
$\forall$	"for all"	all, every, any, each, none, no
$\exists$	"there exists"	there is, some, at least one, not all

- **Universal**: $\forall x \in D, P(x)$ — "for all x in D, $P(x)$".
- **Existential**: $\exists x \in D$ such that $P(x)$ — "there exists an x in D such that $P(x)$". "such that" is commonly abbreviated **s.t.**

Note the punctuation convention: $\forall$ takes a comma, $\exists$ takes "such that".

Translating English to symbols

English	Symbolic
Every rational number is a real number.	$\forall x \in \mathbb{Q}, x \in \mathbb{R}$
No integer is an irrational number.	$\forall x \in \mathbb{Z}, x \notin \overline{\mathbb{Q}}$
Some real numbers are rational.	$\exists x \in \mathbb{R}$ s.t. $x \in \mathbb{Q}$
Not all rational numbers are integers.	$\exists x \in \mathbb{Q}$ s.t. $x \notin \mathbb{Z}$
All prime numbers are odd.	$\forall x \in P, x$ is odd (with P the set of primes)
If an integer is divisible by 2 then it is divisible by 4.	$\forall x \in \mathbb{Z}$, if $2 \mid x$ then $4 \mid x$

Two things to watch:

- **"No ..." is universal, not existential** — "no integer is irrational" becomes $\forall$, with the predicate negated.
- **Sometimes the quantifier is implicit.** "If an integer is a square, then it is nonnegative" carries a hidden $\forall$: $\forall n \in \mathbb{Z}$, if n is a square then $n \geq 0$.

Determining truth

- $\forall x \in D, P(x)$ is shown **false** by one counterexample.
- $\exists x \in D$ s.t. $P(x)$ is shown **true** by one example.

The asymmetry matters: a single value can settle a universal statement only in the negative, and an existential statement only in the positive.

$\exists y \in \mathbb{Z}$ s.t. $y^2 - y = 0$ is **true** — choose $y = 0$ or $y = 1$. That some other choice ($y = 2$) fails is irrelevant; $\exists$ only asks for one witness. The corresponding universal statement $\forall y \in \mathbb{Z}, y^2 - y = 0$ is false, and $y = 2$ is exactly the counterexample that shows it.

$\forall x \in \mathbb{Z}$, if 6 is divisible by x then $x = 2$ is **false** — take $x = 3$: it's an integer, 6 is divisible by 3, but $3 \neq 2$. (The corrected true statement is $\forall x \in \mathbb{Z}$, if 6 is divisible by x then $x = \pm 1, \pm 2, \pm 3$ or ± 6.)

Negation

Negating a quantified statement **flips the quantifier and negates the predicate**:

$$\sim (\forall x \in D, P(x)) \equiv \exists x \in D \text{ s.t. } \sim P(x)$$

$$\sim (\exists x \in D \text{ s.t. } P(x)) \equiv \forall x \in D, \sim P(x)$$

If the negation of a statement is true, the original is false — so proving the negation is a legitimate way to disprove a statement (see proof-techniques[149]).

When the predicate is itself conditional, negate it with $\sim (p \to q) \equiv p \wedge \sim q$ (see logical-equivalence-laws[150]). So the negation of

$$\forall x \in \mathbb{Z}, \text{ if } x^2 > 3 \text{ then } x > 3$$

is $\exists x \in \mathbb{Z}$ s.t. $x^2 > 3$ **and** $x \leq 3$ — *not* "if … then …" with an $\exists$ in front, and not the converse.

A negation is not just "the opposite-sounding statement"

For $S = \{1, 3, 5, 7\}$, $\{1, 2, 3, 4\}$, $\{2, 4, 6, 8\}$:

S	$\forall x \in S, x$ is odd	$\forall x \in S, x$ is not odd	$\exists x \in S$ s.t. x is not odd
$\{1, 3, 5, 7\}$	T	F	F
$\{1, 2, 3, 4\}$	F	F	T
$\{2, 4, 6, 8\}$	F	T	T

The first and *third* columns are exact negations of each other (they disagree in every row). The middle column — "all are not odd" — is **not** the negation: for $S = \{1, 2, 3, 4\}$ both it and the original are false.

[149] courses/math1061/proof-techniques.pdf
[150] courses/math1061/logical-equivalence-laws.pdf

Worked negations

Statement	Negation	Which is true
$\exists y \in \mathbb{R}$ s.t. $y^2 < 0$	$\forall y \in \mathbb{R}, y^2 \geq 0$	negation
All dogs are white.	Some dog is not white.	negation
$\forall x \in \mathbb{Z}^+, x$ is prime or x is composite	$\exists x \in \mathbb{Z}^+$ s.t. x is neither prime nor composite	negation ($x = 1$)
$\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$	$\exists x \in \mathbb{R}$ s.t. $x > 2$ and $x^2 \leq 4$	original
$\forall x \in \mathbb{R}$, if $x^2 > 9$ then $x \geq 2$	$\exists x \in \mathbb{R}$ s.t. $x^2 > 9$ and $x < 2$	negation ($x = -4$)

Multiple quantifiers

Order matters. Read the statement left to right as a game:

- $\forall x \in X, \exists y \in Y$ s.t. $P(x, y)$ — "someone gives me an x; can I *always* then find a y?" The choice of y may depend on x.
- $\exists x \in X$ s.t. $\forall y \in Y, P(x, y)$ — "is there a *single* x that works for *every* y?" The x must be chosen before seeing y.

Both of these happen to be true for $xy \geq y$ over $\mathbb{Z}$, but for different reasons:

1. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $xy \geq y$ — given x, choose $y = x$ (since $x^2 \geq x$ for every integer), or more simply $y = 0$.
2. $\exists x \in \mathbb{Z}$ s.t. $\forall y \in \mathbb{Z}, xy \geq y$ — take $x = 1$, since $y \geq y$ always.

Where order really bites: $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $y = 2x$ is **true** (given x, take $y = 2x$), but $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $x = 2y$ is **false** — take $x = 3$, and $y = \frac{3}{2} \notin \mathbb{Z}$.

Writing two of the same quantifier

There are three forms:

- $\exists x \in D$ and $\exists y \in E \ldots$ — the clearest.
- $\exists x \in D$ and $y \in E \ldots$
- $\exists x, y \in D \ldots$ — shorthand for the same domain, only correct when both variables share a domain.

All three are acceptable; the first is the safe one.

Negating multiple quantifiers

Flip **every** quantifier in order, and negate the predicate at the end:

$$\sim (\forall x \in X, \exists y \in Y \text{ s.t. } P(x,y)) \equiv \exists x \in X \text{ s.t. } \forall y \in Y, \sim P(x,y)$$

The **order of the variables never moves** — only the quantifiers themselves change. A colour-coded, part-for-part version of this exchange is drawn in 2026-08-06-negation-and-multiple-quantifiers[151].

Worked examples:

Statement	Negation	Which is true
$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x + y = 0$	$\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, x + y \neq 0$	original (take $y = -x$)
$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy = 1$	$\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, xy \neq 1$	negation ($x = 0$)
$\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, xy = 0$	$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy \neq 0$	original (take $x = 0$)
$\exists x \in \mathbb{Z}^+$ s.t. $\forall y \in \mathbb{Z}^+, x \geq y$	$\forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+$ s.t. $x < y$	negation (take $y = x + 1$)
$\forall x, y \in \mathbb{Z}, \exists z \in \mathbb{Z}$ s.t. $z = x - y$	$\exists x, y \in \mathbb{Z}$ s.t. $\forall z \in \mathbb{Z}, z \neq x - y$	original
$\forall r, s \in \mathbb{R}$, if $rs \in \mathbb{Q}$ then $r \in \mathbb{Q}$ and $s \in \mathbb{Q}$	$\exists r, s \in \mathbb{R}$ s.t. $rs \in \mathbb{Q}$ and ($r \notin \mathbb{Q}$ or $s \notin \mathbb{Q}$)	negation ($r = s = \sqrt{2}$)

A negation trap: the negation of "$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $x + y$ is odd" is "**there exists** an integer x such that **for every** integer y, $x + y$ is even". (The original is true: if x is odd take $y = 0$, if x is even take $y = 1$.)

See also

- logical-connectives[152] · logical-equivalence-laws[153] · conditional-statements[154]
- proof-techniques[155] — proving and disproving quantified statements
- practice-problems[156] — §6 Quantified Statements, with full worked solutions
- practice-problems[157] — §7 Negation of Quantified Statements, with full worked solutions
- practice-problems[158] — §8 Statements with Multiple Quantifiers, with full worked solutions

[151] courses/math1061/2026-08-06-negation-and-multiple-quantifiers.pdf
[152] courses/math1061/logical-connectives.pdf
[153] courses/math1061/logical-equivalence-laws.pdf
[154] courses/math1061/conditional-statements.pdf
[155] courses/math1061/proof-techniques.pdf
[156] courses/math1061/practice-problems.pdf
[157] courses/math1061/practice-problems.pdf
[158] courses/math1061/practice-problems.pdf

Rational & Irrational Numbers

The definitions and closure results that make $\mathbb{Q}$ a workable object in proofs, plus the classic irrationality argument.

Definitions

$$r \in \mathbb{Q} \iff \exists a, b \in \mathbb{Z} \text{ such that } r = \frac{a}{b} \text{ and } b \neq 0$$

$$r \in \overline{\mathbb{Q}} \iff r \notin \mathbb{Q}$$

$\overline{\mathbb{Q}} = \mathbb{R} \setminus \mathbb{Q}$ is the set of **irrational** numbers. The barred symbol is what this course writes, not $\mathbb{R} \setminus \mathbb{Q}$. Figure 14 shows how the four sets sit together.

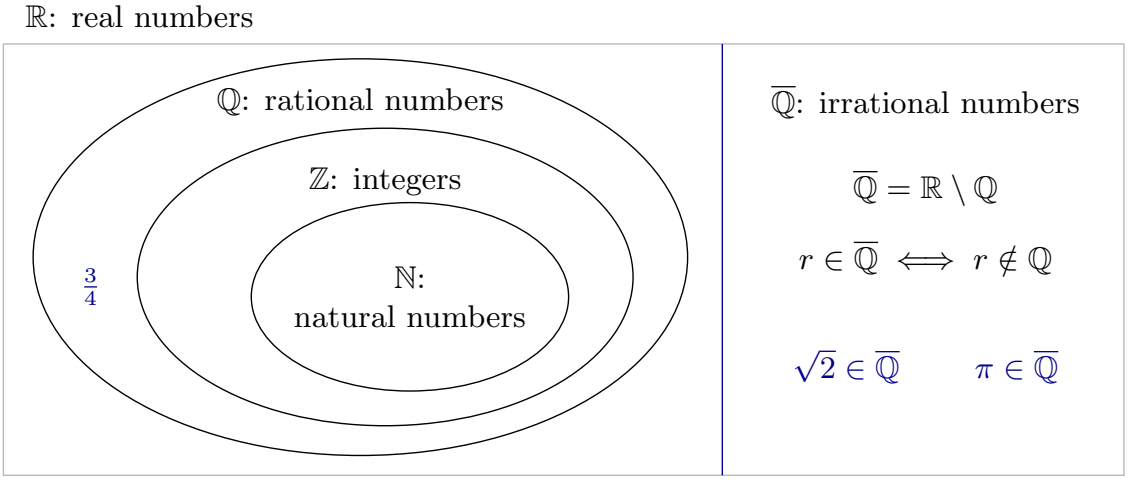


Figure 14: The number systems as nested closed curves with each name written inside its own region, and the irrationals in their own panel because they are the part of the reals that is left over rather than a set that contains any of the others. The three examples are in blue. Two things to read off it — every integer is rational (it is n/1), and no number is in both panels, which is exactly what the definition of the bar says. The course leaves open whether N contains 0, so nothing here should be read as settling that.

Note the definition is existential: to show r is rational you must **exhibit** integers a and b with $b \neq 0$. Nearly every proof below ends by checking three things — the numerator is an integer, the denominator is an integer, and the denominator is non-zero.

Fact: every rational number can be written as a fraction in **lowest terms** (numerator and denominator have no common factors).

Why irrationality statements want contradiction or contraposition

"r is irrational" is a *negative* statement — $r \notin \mathbb{Q}$ — so a direct proof has nothing to unfold. Assuming the negation instead ($r \in \mathbb{Q}$) hands you $r = \frac{a}{b}$ to work with. See proof-techniques[159] for the general shape, and the collected verdicts in 2026-08-13-proof-by-contraposition[160] for which method works on each statement.

Concretely, for **the sum of any rational number and any irrational number is irrational**:

- Symbolically: $\forall r, s \in \mathbb{R}$, if $r \in \mathbb{Q}$ and $s \in \overline{\mathbb{Q}}$ then $r + s \in \overline{\mathbb{Q}}$ — or equivalently $\forall r \in \mathbb{Q}$ and $\forall s \in \mathbb{R}$, if $s \in \overline{\mathbb{Q}}$ then $r + s \in \overline{\mathbb{Q}}$.
- **Direct proof**: suppose $r \in \mathbb{Q}$ and $s \in \overline{\mathbb{Q}}$ — and now you're stuck, because $s \in \overline{\mathbb{Q}}$ gives you nothing to substitute.
- **Contradiction**: suppose $\exists r \in \mathbb{Q}, s \in \mathbb{R}$ with $s \in \overline{\mathbb{Q}}$ and $r + s \in \mathbb{Q}$.
- **Contraposition**: prove $\forall r \in \mathbb{Q}$ and $\forall s \in \mathbb{R}$, if $r + s \in \mathbb{Q}$ then $s \in \mathbb{Q}$.

  Proof by contraposition. Suppose $r \in \mathbb{Q}$, $s \in \mathbb{R}$ and $r + s \in \mathbb{Q}$. Since $r \in \mathbb{Q}$, $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Since $r + s \in \mathbb{Q}$, $r + s = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$. Now
  $$s = (r + s) - r = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}.$$
  Since $a, b, c, d \in \mathbb{Z}$ we know $bc - ad \in \mathbb{Z}$ and $bd \in \mathbb{Z}$; and $bd \neq 0$ since $b \neq 0$ and $d \neq 0$. Thus $s \in \mathbb{Q}$. $\square$

The trick that makes it work is writing the thing you want in terms of the things you have: $s = (r + s) - r$. The second fraction needs a **fresh pair of variables** c, d — nothing relates it to a, b, and reusing them is one of the standard errors in proof-techniques[161].

Closure results

$\forall r \in \mathbb{Q}, (2r^2 - 3r + 1) \in \mathbb{Q}$ — direct proof is best here; contraposition gets stuck.

Let $r \in \mathbb{Q}$, so $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Then
$$2r^2 - 3r + 1 = \frac{2a^2}{b^2} - \frac{3ab}{b^2} + \frac{b^2}{b^2} = \frac{2a^2 - 3ab + b^2}{b^2}.$$

[159] courses/math1061/proof-techniques.pdf
[160] courses/math1061/2026-08-13-proof-by-contraposition.pdf
[161] courses/math1061/proof-techniques.pdf

Since $a, b \in \mathbb{Z}$ we have $2a^2 - 3ab + b^2 \in \mathbb{Z}$ and $b^2 \in \mathbb{Z}$, and $b^2 \neq 0$ since $b \neq 0$. So $2r^2 - 3r + 1 \in \mathbb{Q}$. □

$\forall r \in \mathbb{R}$, if r^2 is irrational then r is irrational. Proved by contraposition in proof-techniques[162] — the contrapositive is "if $r \in \mathbb{Q}$ then $r^2 \in \mathbb{Q}$", i.e. $r^2 = \frac{a^2}{b^2}$ with $a^2, b^2 \in \mathbb{Z}$ and $b^2 \neq 0$.

For real m, n: if m is irrational and n is rational, then $m - n$ is irrational.

Proof by contradiction. Suppose m is irrational, n is rational, but $m - n$ is rational. Then $n = \frac{a}{b}$ and $m - n = \frac{c}{d}$ for integers a, b, c, d with $b, d \neq 0$. Since $m = (m-n)+n$,

$$m = \frac{c}{d} + \frac{a}{b} = \frac{bc + ad}{bd},$$

with $bc + ad \in \mathbb{Z}$, $bd \in \mathbb{Z}$ and $bd \neq 0$. So m is rational, contradicting the assumption. □

For a real number r: if $2r^2 - 3r$ is irrational, then r is irrational.

Proof by contraposition. The contrapositive is: if $r \in \mathbb{Q}$ then $2r^2 - 3r \in \mathbb{Q}$. Let $r = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$. Then $2r^2 - 3r = \frac{2a^2 - 3ab}{b^2}$, with integer numerator and non-zero integer denominator. □

For all real r, s and n: if r is irrational, s is rational and n is a positive integer, then $n(r - s)$ is irrational.

Proof by contradiction. Suppose otherwise: $r \in \overline{\mathbb{Q}}$, $s \in \mathbb{Q}$, $n \in \mathbb{Z}^+$, but $n(r - s) \in \mathbb{Q}$. Write $s = \frac{a}{b}$ and $n(r - s) = \frac{c}{d}$ with $b, d \neq 0$. Since $n \neq 0$,

$$r = s + \frac{c}{d} \cdot \frac{1}{n} = \frac{a}{b} + \frac{c}{dn} = \frac{adn + bc}{bdn}.$$

Here $adn + bc \in \mathbb{Z}$, $bdn \in \mathbb{Z}$, and $bdn \neq 0$ since b, d, n are all non-zero. So $r \in \mathbb{Q}$ — a contradiction. □

For every rational r, if $r \neq 0$ then $r\sqrt{2}$ is irrational. Negation: $\exists r \in \mathbb{Q}$ such that $r \neq 0$ and $r\sqrt{2} \in \mathbb{Q}$.

Proof by contradiction. Suppose the negation holds. Since $r \in \mathbb{Q}$, $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$, and $a \neq 0$ because $r \neq 0$. Since $r\sqrt{2} \in \mathbb{Q}$, $r\sqrt{2} = \frac{c}{d}$ for integers c, d with $d \neq 0$. Then

$$\sqrt{2} = \frac{r\sqrt{2}}{r} = \frac{c}{d} \cdot \frac{b}{a} = \frac{bc}{ad},$$

where $bc \in \mathbb{Z}$, $ad \in \mathbb{Z}$ and $ad \neq 0$ (since $a \neq 0$ and $d \neq 0$). So $\sqrt{2} \in \mathbb{Q}$ — contradicting the theorem below. □

[162] courses/math1061/proof-techniques.pdf

$\sqrt{2}$ is irrational

The set-up needs two ingredients: the lowest-terms fact above, and the lemma **for all integers n, if n^2 is even then n is even** (proved by contraposition — see proof-techniques[163]).

> *Proof by contradiction.* Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$, and we may assume $\frac{a}{b}$ is in lowest terms.
>
> Squaring, $2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$ and thus a^2 is even. By the lemma, a is even, so $a = 2\ell$ for some $\ell \in \mathbb{Z}$.
>
> Substituting: $2b^2 = a^2 = (2\ell)^2 = 4\ell^2$, so $b^2 = 2\ell^2$ and b^2 is even. By the lemma again, b is even.
>
> But now a and b are both even, so they share the common factor 2 — contradicting the assumption that $\frac{a}{b}$ is in lowest terms. Therefore $\sqrt{2} \in \overline{\mathbb{Q}}$. $\square$

The shape of the argument is easier to hold onto than its algebra (Figure 15).

Neither a direct proof nor a contraposition proof gets off the ground here: the statement isn't a conditional with a usable hypothesis, so contradiction is the only one of the three that has anywhere to start.

See also

- proof-techniques[164] — direct / counterexample / contradiction / contraposition
- 2026-08-13-proof-by-contraposition[165] — Lecture 9
- quantified-statements[166]
- divisibility-and-factorisation[167]
- practice-problems[168] — §12 Rational Numbers, with full worked solutions

[163] courses/math1061/proof-techniques.pdf
[164] courses/math1061/proof-techniques.pdf
[165] courses/math1061/2026-08-13-proof-by-contraposition.pdf
[166] courses/math1061/quantified-statements.pdf
[167] courses/math1061/divisibility-and-factorisation.pdf
[168] courses/math1061/practice-problems.pdf

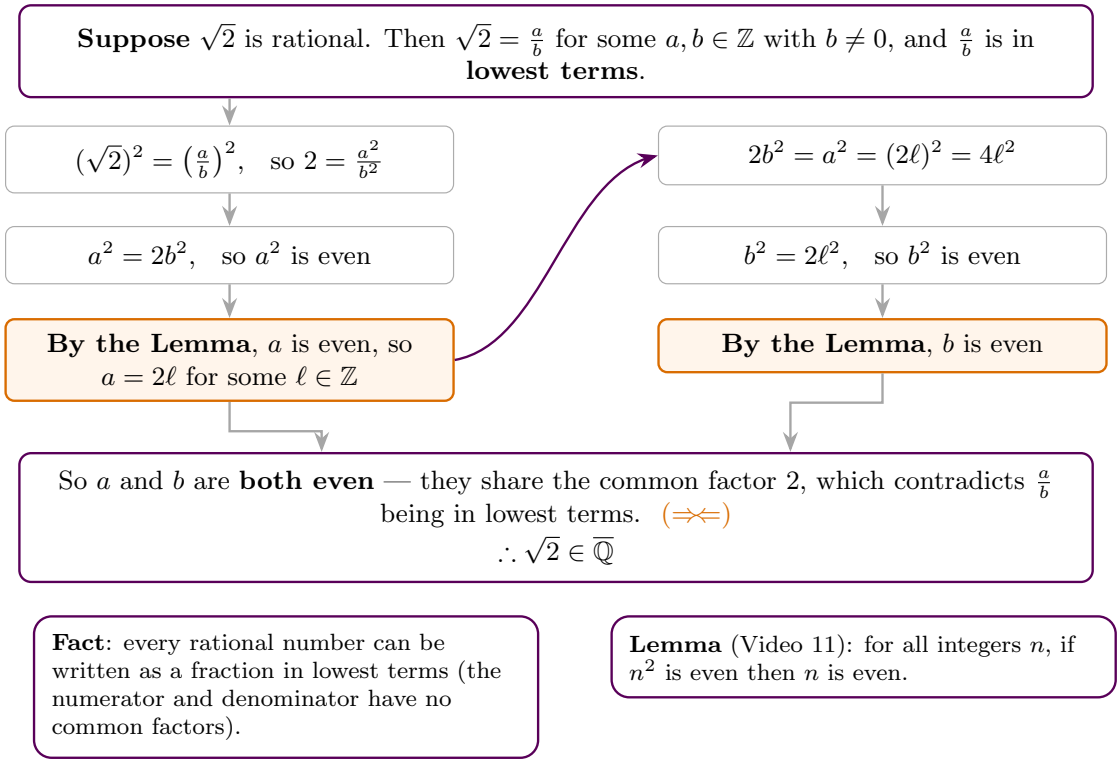


Figure 15: The same proof as a structure. It runs in two halves that use the lemma once each (orange): the left half squeezes 'a is even' out of the assumption, and the substitution a = 2l feeds the right half, which gets 'b is even'. Neither half is a contradiction on its own — the contradiction only appears when both land together and the fraction turns out not to have been in lowest terms after all. That is why the lowest-terms assumption has to be made in the very first line: it is the thing the two halves eventually collide with. The two boxes at the bottom state the Fact and the Lemma.