Modular Arithmetic, Floor & the Quotient-Remainder Theorem

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Floor and ceiling

For a real number x:

- $\lfloor x \rfloor$ — the **floor**: move **left** on the number line to the nearest integer.
- $\lceil x \rceil$ — the **ceiling**: move **right** on the number line to the nearest integer.

So $\lfloor -3.2 \rfloor = -4$ and $\lceil -3.2 \rceil = -3$; $\lfloor 3.2 \rfloor = 3$ and $\lceil 3.2 \rceil = 4$. **Negative numbers are where this goes wrong most often** — floor of -3.2 is -4, not -3.

To use floor or ceiling in a proof, it is usually easiest to rewrite the expression as **an integer plus an amount between 0 and 1**. For instance, in proving "if n is an odd integer then $\lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor + 1$", write

$$\frac{n}{2} = \frac{2k+1}{2} = k + \frac{1}{2}$$

for some integer k; then $k + \frac{1}{2}$ sits strictly between k and $k+1$, so the floor is k and the ceiling is $k+1$. The same move handles things like $\lfloor (p^3 + 4p^2 + 5) + \frac{1}{4} \rfloor$ where $p \in \mathbb{Z}$.

Two ways to compute $\lfloor \frac{-32}{5} \rfloor$:

- *Fraction approach*: $\lfloor \frac{-32}{5} \rfloor = \lfloor \frac{-30}{5} - \frac{2}{5} \rfloor = \lfloor -6 - \frac{2}{5} \rfloor = -7$ (something smaller than -6).
- *Decimal approach*: $\lfloor -6.4 \rfloor = \lfloor -6 - 0.4 \rfloor = -7$.

And $\lceil \frac{-32}{5} \rceil = -6$.

$\forall x \in \mathbb{R}, \lfloor x^2 \rfloor = \lfloor x \rfloor^2$ **is false** — take $x = \frac{3}{2}$: $\lfloor \frac{9}{4} \rfloor = 2$ but $\lfloor \frac{3}{2} \rfloor^2 = 1^2 = 1$.

The quotient-remainder theorem

Given any integer n and positive integer d, there exist **unique** integers q and r such that

$$n = dq + r \quad \text{and} \quad 0 \leq r < d$$

q is the **quotient** ($n \operatorname{div} d$) and r is the **remainder** ($n \bmod d$). Explicitly:

$$q = \left\lfloor \frac{n}{d} \right\rfloor \qquad r = n - d \left\lfloor \frac{n}{d} \right\rfloor$$

With $d = 6$:

n	$n = dq + r$	q	r
53	$6 \cdot 8 + 5$	8	5
20	$6 \cdot 3 + 2$	3	2
-19	$6 \cdot (-4) + 5$	-4	5

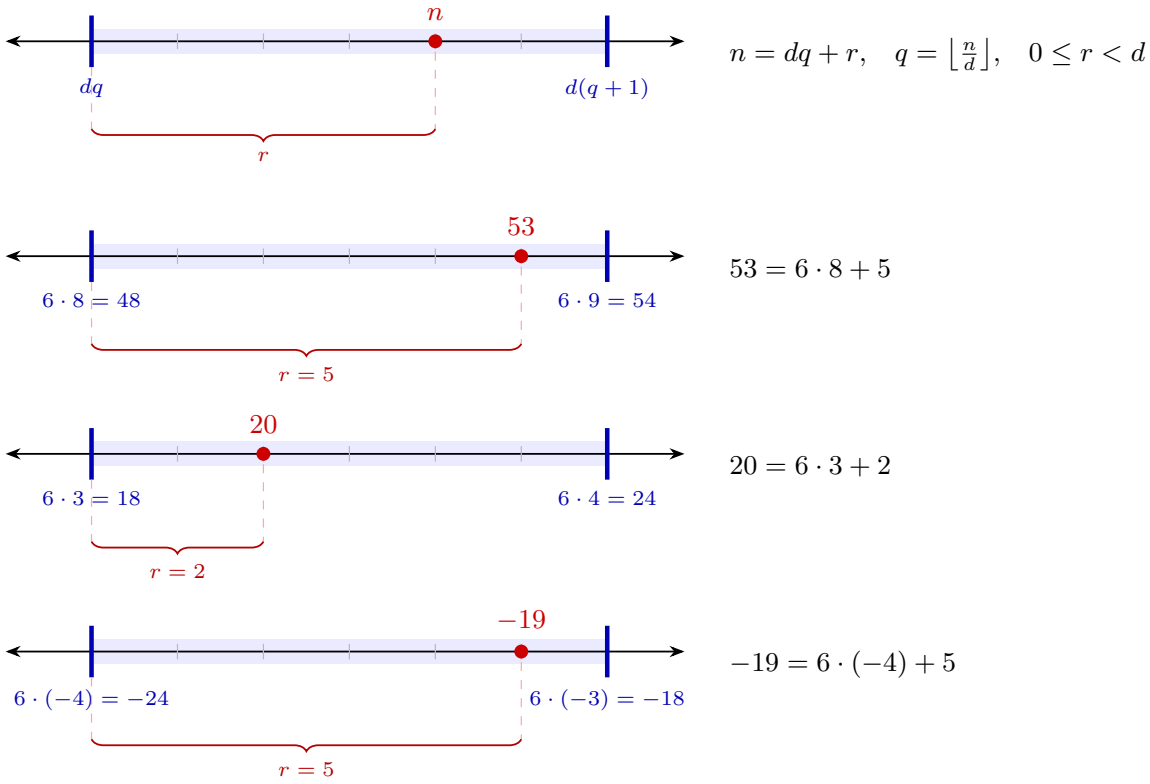
Negative n is the case to watch: the remainder is still required to be non-negative, so $-19 \bmod 6 = 5$, not -1. Likewise $-95 = 11(-9) + 4$, so $-95 \operatorname{div} 11 = -9$ and $-95 \bmod 11 = 4$. And $58 = 11 \cdot 5 + 3$, so $58 \operatorname{div} 11 = 5$, $58 \bmod 11 = 3$.

mod versus $\equiv$ — take care with notation

These are different statements.

- $n \bmod d = r$ — an *equality between numbers*: the remainder when n is divided by d is r, with $0 \leq r < d$.
- $m \equiv n \pmod{d}$ — a *relation between m and n*: they leave the same remainder when divided by d.

Example: $5 \bmod 4 = 1$ and $9 \bmod 4 = 1$, so $9 \equiv 5 \pmod 4$ — but $9 \bmod 4 \neq 5$.



The shaded band is the half-open block $[\,dq,\ d(q+1)\,)$ that n lands in. The quotient q names the multiple of d at or *below* n, so the remainder r is always measured rightwards — which is why -19 has remainder 5 and not -1.

Figure 1: Why the remainder is never negative: the general case and the three $d = 6$ examples on the number line. Each shaded band is one block $[dq, d(q+1))$ of width d; the quotient names the multiple of d at or below n, so r is always the rightward gap from that multiple, and -19 therefore has remainder 5 rather than -1.

Two equivalent definitions of congruence

For $m, n, d \in \mathbb{Z}$ with $d > 0$:

- **Definition 1**: $m \equiv n \pmod{d}$ iff m and n have the same remainder under the quotient-remainder theorem.
- **Definition 2**: $m \equiv n \pmod{d}$ iff $d \mid (m - n)$.

Definition 2 is usually the faster one to check, and the one to reach for in proofs (see divisibility-and-factorisation[1]).

Worked both ways — is $-50 \equiv 22 \pmod{8}$?

- *Definition 1*: $-50 = -56 + 6 = 8(-7) + 6$, so $-50 \bmod 8 = 6$; and $22 = 16 + 6$, so $22 \bmod 8 = 6$. Same remainder, so yes.
- *Definition 2*: $22 - (-50) = 72 = 8 \cdot 9$, so $8 \mid (22 - (-50))$, so yes.

Similarly $53 \equiv -19 \pmod{6}$ (both leave remainder 5; and $53 - (-19) = 72$ is divisible by 6), while $53 \not\equiv 20 \pmod{6}$.

Arithmetic with congruences

Lemma: if $a \equiv b \pmod{d}$ and $m \equiv n \pmod{d}$, then

$$a + m \equiv b + n \pmod{d} \qquad \text{and} \qquad am \equiv bn \pmod{d}$$

That is, you may substitute any number for a congruent one *before* adding or multiplying — which is what makes modular arithmetic cheap.

For example, $8 \equiv 2 \pmod{3}$ and $46 \equiv 1 \pmod{3}$, so $8 \cdot 46 \equiv 2 \cdot 1 \equiv 2 \pmod{3}$ and $8 + 46 \equiv 2 + 1 \equiv 0 \pmod{3}$.

Worked: suppose $x \bmod 6 = 4$ and $y \bmod 6 = 3$.

- $(x + y) \bmod 6$: $x + y \equiv 4 + 3 \equiv 7 \equiv 1 \pmod{6}$, so the answer is 1.
- $(5x^2 + 4y) \bmod 6$: $5x^2 + 4y \equiv 5(4)^2 + 4(3) \equiv 5(16) + 12 \equiv 5(4) \equiv 20 \equiv 2 \pmod{6}$ — applying the lemma at each step.

Worked: if $m \bmod 6 = 4$, what is $35m \bmod 6$? Since $35 \bmod 6 = 5$, $35m \equiv 5 \cdot 4 \equiv 20 \equiv 2 \pmod{6}$. And $3m^2 + 2m + 1 \equiv 3(4)^2 + 2(4) + 1 \equiv 12(4) + 8 + 1 \equiv 8 + 1 \equiv 3 \pmod{6}$ — $3 \cdot 4^2$ is $12 \cdot 4$, and that term drops out whole, since $6 \mid 12$.

Worked: an integer n leaves remainder 4 on division by 7 — what is the remainder of $3n$? Write $n = 7k + 4$; then $3n = 21k + 12 = 7(3k + 1) + 5$, so the remainder is 5. (The course's

[1] courses/math1061/divisibility-and-factorisation.pdf

solutions PDF has this step as $7(3k)+5$, which is $21k+5$, not $21k+12$; the quotient is $3k+1$. The answer of 5 is unaffected — see practice-problems[2].)

Worked: m leaves remainder 9 on division by 11 — what remainder does $4m$ leave on division by **22**? Write $m = 11k+9$; then $4m = 44k+36 = 22(2k+1)+14$, so the remainder is 14. (Note the modulus changed, so you cannot just reduce mod 11.)

Divisibility tests

Modular arithmetic explains the familiar tests. **A positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3**, because $10 \equiv 1 \pmod 3$ and hence $10^k \equiv 1 \pmod 3$ for every $k \in \mathbb{Z}^+$ (apply the multiplication half of the lemma repeatedly: $10^2 \equiv 1$ from $a = m = 10$, $b = n = 1$; then $10^3 \equiv 1$ from $a = 10^2$, $m = 10$; and so on).

$$254 \equiv 2 \cdot 100 + 5 \cdot 10 + 4 \cdot 1 \equiv 2 + 5 + 4 \equiv 11 \equiv 2 \pmod 3$$

so 254 is not divisible by 3 and leaves remainder 2 (checking: $254 = 3 \cdot 84 + 2$). Whereas

$$3252 \equiv 3 + 2 + 5 + 2 \equiv 12 \equiv 0 \pmod 3$$

so $3 \mid 3252$.

Worked proofs

For any odd integer n, $n^2 \bmod 4 = 1$.

> Suppose n is odd, so $n = 2k+1$ for some $k \in \mathbb{Z}$. Then
>
> $$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2+k) + 1.$$
>
> Since $k^2 + k \in \mathbb{Z}$ and $0 \le 1 < 4$, the quotient-remainder theorem gives $n^2 \bmod 4 = 1$. □

Hence for any odd integer n, $\left\lfloor \frac{n^2}{4} \right\rfloor = \frac{n^2-1}{4}$. Two routes:

- *Method 1 — expand both sides.* With $n = 2k+1$: LHS $= \lfloor \frac{4k^2+4k+1}{4} \rfloor = \lfloor (k^2+k) + \frac{1}{4} \rfloor = k^2 + k$. RHS $= \frac{(2k+1)^2-1}{4} = \frac{4k^2+4k}{4} = k^2 + k$. Equal.
- *Method 2 — use $r = n - d\lfloor \frac{n}{d} \rfloor$.* With number n^2, divisor 4, remainder 1: $1 = n^2 - 4\lfloor \frac{n^2}{4} \rfloor$, so $4\lfloor \frac{n^2}{4} \rfloor = n^2 - 1$ and $\lfloor \frac{n^2}{4} \rfloor = \frac{n^2-1}{4}$.

[2] courses/math1061/practice-problems.pdf

For all odd integers n, $\left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)$ — true, since with $n = 2k+1$ the right-hand side is $\frac{2k}{2} \cdot \frac{2k+2}{2} = k(k+1)$, matching the $k^2 + k$ above.

If y is any integer, then y^2 divided by 5 leaves remainder 0, 1 or 4 — never 2 or 3.

> Fix $y \in \mathbb{Z}$. By the quotient-remainder theorem, $y = 5m + r$ for some $m \in \mathbb{Z}$ and $r \in \{0, 1, 2, 3, 4\}$. Then
>
> $$y^2 = 25m^2 + 10mr + r^2 \equiv r^2 \pmod 5.$$
>
> Checking each case: $0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 4$, $3^2 \equiv 9 \equiv 4$, $4^2 \equiv 16 \equiv 1 \pmod 5$. So $y^2 \equiv 0, 1$ or $4 \pmod 5$. □

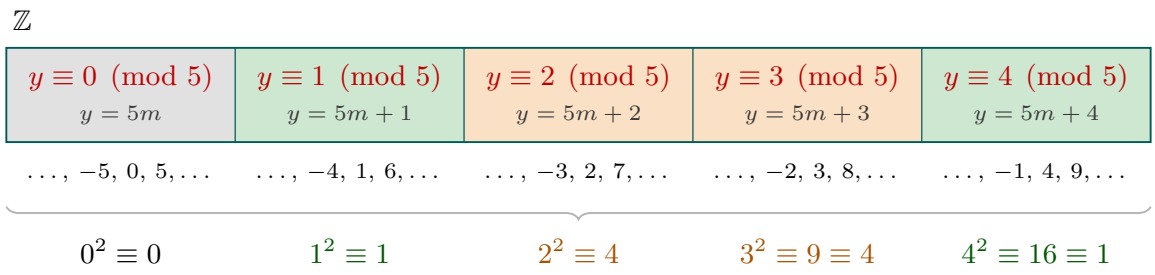


Figure 2: The case split the proof above runs on: $\mathbb{Z}$ cut into the five residue classes mod 5. The tints group the classes by the remainder their square leaves, which is what makes 2 and 3 unreachable.

This "split into cases by remainder" move is the standard way to prove something about *every* integer using modular arithmetic.

See also

- divisibility-and-factorisation[3] · gcd-lcm-and-euclidean-algorithm[4] · proof-techniques[5]
- sets-and-set-notation[6] — congruence classes as sets, and the partitions of $\mathbb{Z}$ they give
- equivalence-relations[7] · integers-modulo-n[8] — congruence as an equivalence relation, and its classes as the number system $\mathbb{Z}_n$

[3] courses/math1061/divisibility-and-factorisation.pdf
[4] courses/math1061/gcd-lcm-and-euclidean-algorithm.pdf
[5] courses/math1061/proof-techniques.pdf
[6] courses/math1061/sets-and-set-notation.pdf
[7] courses/math1061/equivalence-relations.pdf
[8] courses/math1061/integers-modulo-n.pdf

- pigeonhole-principle[9] — the n remainders as pigeonholes: among any $n+1$ integers, two differ by a multiple of n
- practice-problems[10] — §14 Modular Arithmetic, with full worked solutions

[9] courses/math1061/pigeonhole-principle.pdf
[10] courses/math1061/practice-problems.pdf