Conditional & Biconditional Statements

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Conditional ($\to$)

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Only one row makes a conditional false: the row where the hypothesis p is true and the conclusion q is false. Everything else is true, including both rows where p is false.

$p \to q$ is true when p is false **because that is how it is defined.** There is no deeper intuition to recover; $\to$ is a truth-functional connective fixed by the table above, and arguing with the bottom two rows is arguing with a definition. The standard name for those two rows is **vacuous truth**. It does real work later: it is why $\emptyset \subseteq A$ for every set A (see sets-and-set-notation[1]).

The picture behind the table is containment:

[1] courses/math1061/sets-and-set-notation.pdf

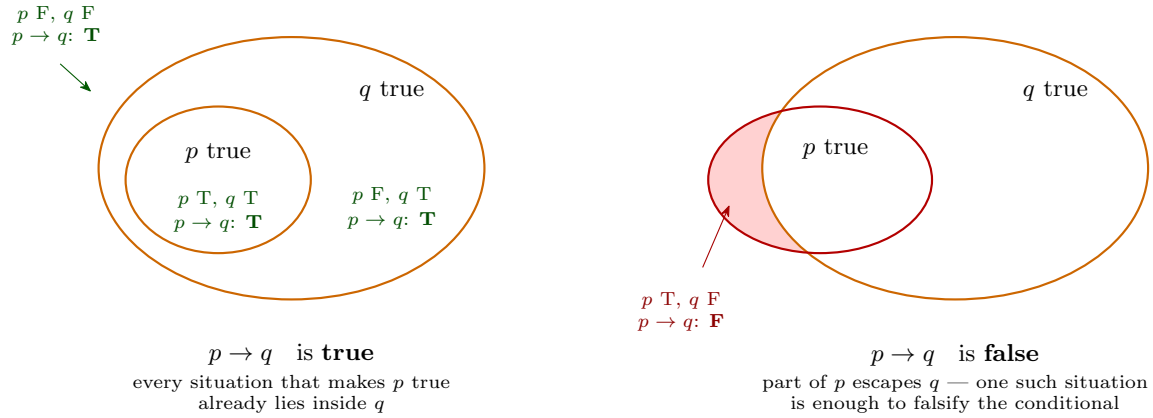

Figure 1: Why the truth table has the shape it does. Left: p → q asserts that the situations making p true all lie inside the situations making q true, so every region carries T. Right: the conditional fails exactly when part of p escapes q, which is the single F row of the table.

Disjunctive form

By definition,

$$p \to q \equiv \sim p \vee q$$

The right-hand side is the **disjunctive form** of the conditional, and it is the single most useful fact in the topic: the laws of logical equivalence (see logical-equivalence-laws[2]) are stated only for $\sim$, $\wedge$ and $\vee$, so they say nothing about $\to$ directly. To use the laws on a conditional:

> **Replace $p \to q$ with $\sim p \vee q$ and then use the laws.**

Every worked proof about conditionals in this course starts and (usually) ends with that substitution.

$p \to q$ can be phrased: "p implies q", "if p then q", "q if p", "p is sufficient for q", "p only if q", "q is necessary for p".

Warning: $p \to q \not\equiv q \to p$.

Contrapositive, converse, inverse

[2] courses/math1061/logical-equivalence-laws.pdf

Name	Form	Equivalent to
Conditional	$p \to q$	Contrapositive
Contrapositive	$\sim q \to \sim p$	Conditional
Converse	$q \to p$	Inverse
Inverse	$\sim p \to \sim q$	Converse

The conditional and its **contrapositive** are always logically equivalent. The **converse** and **inverse** are always logically equivalent to each other — but *not* to the original conditional.

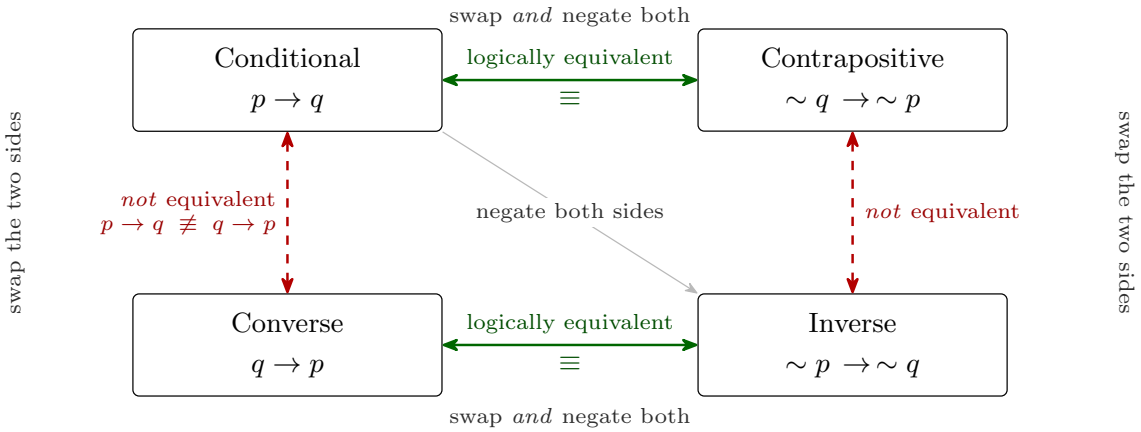

Figure 2: The four forms and the only equivalences among them. The conditional and the contrapositive are one logically equivalent pair, the converse and the inverse are the other, and p → q q → p. Swapping the two sides gives the converse, negating both gives the inverse, and doing both gives the contrapositive — the only one of the three that preserves meaning.

Both equivalences fall straight out of the disjunctive form. For the first: $p \to q \equiv \sim p \vee q$ and $\sim q \to \sim p \equiv \sim(\sim q) \vee \sim p \equiv q \vee \sim p$, which is the same disjunction commuted. The converse/inverse pair is the same computation with p and q interchanged.

Necessary and sufficient conditions

- p is a **necessary condition** for q means "if $\sim p$ then $\sim q$", or equivalently "if q then p", or "q **only if** p". In symbols, $q \to p$.
- p is a **sufficient condition** for q means "if p then q", or equivalently "q **if** p". In symbols, $p \to q$.

The two phrasings of "necessary" are contrapositives of each other, which is why they say the same thing: $\sim p \rightarrow \sim q$ and $q \rightarrow p$.

The reliable reading is: **"if" introduces the sufficient condition; "only if" introduces the necessary one.**

Worked: necessary but not sufficient

Being a non-negative integer is necessary for a number to be the square of an integer, but it is not sufficient.

- *Necessary*: if a number is the square of an integer, then it is non-negative. So -9 cannot be the square of an integer — failing the condition rules the number out.
- *Not sufficient*: 7 is a non-negative integer but it is not the square of an integer. Meeting the condition does not settle it.

Worked: sufficient but not necessary

Being an integer that is divisible by 4 is sufficient for being an even integer, but it is not necessary.

- *Sufficient*: if an integer is divisible by 4, then it is even. 8 is divisible by 4, so we know 8 is even without checking anything else.
- *Not necessary*: 10 is an even integer but 10 is not divisible by 4. The condition can fail while the conclusion still holds.

The two shapes are exactly opposite in how they are refuted: a claimed **sufficient** condition is killed by something satisfying the condition but not the conclusion; a claimed **necessary** condition is killed by something satisfying the conclusion but not the condition.

Biconditional ($\leftrightarrow$)

$$a \leftrightarrow b \equiv (a \rightarrow b) \wedge (b \rightarrow a)$$

A biconditional is the two one-way conditionals conjoined, and each of the standard English phrasings splits along the same seam:

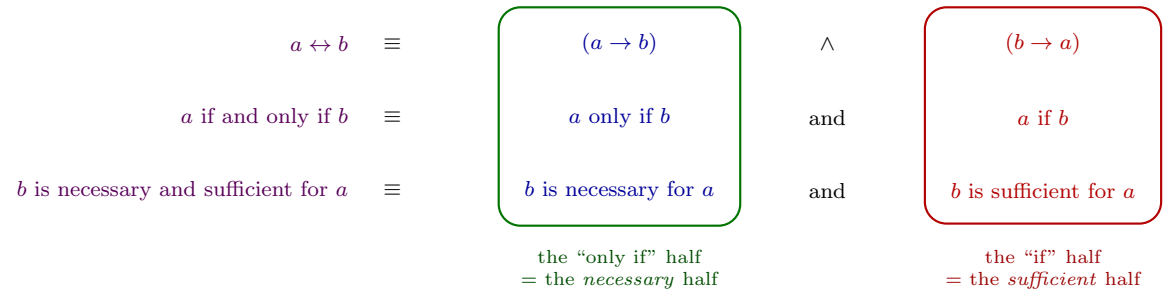

Figure 3: The biconditional decomposed three ways, with each half in its own ringed column. Reading down the left column: a → b, 'a only if b', 'b is necessary for a' are three names for the same half. Reading down the right: b → a, 'a if b', 'b is sufficient for a'. The whole necessary/sufficient vocabulary is just this correspondence — 'only if' is the necessary half, 'if' is the sufficient half.

Negating a conditional

$$\sim (p \to q) \equiv \sim (\sim p \vee q) \equiv p \wedge \sim q$$

by the definition of →, then De Morgan's law, then the double negative law.

The negation of a conditional is not a conditional — it's an *and*. "If it rains, the match is cancelled" is negated by "it rains **and** the match is not cancelled", not by "if it rains, the match is not cancelled".

The classic error is to answer $\sim p \to \sim q$ (that's the *inverse*) or $q \to p$ (the *converse*). Neither is the negation, and neither is even equivalent to the original — see Figure 2 above.

Worked: $\sim (p \to q) \not\equiv \sim p \to \sim q$**.** Let p be **true** and q be **true**. Then $p \to q$ is true, so $\sim (p \to q)$ is **false**. But $\sim p \to \sim q$ is F → F, which is **true**. The two differ on this row, so they are not equivalent.

Sanity check on the shapes: $\sim (p \to q)$ is a conjunction and is true on exactly **one** of the four rows; $\sim p \to \sim q$ is a conditional and is true on **three**. They could hardly be equivalent.

Order of operations

1. $\sim$
2. $\wedge, \vee$ (equal)
3. $\to, \leftrightarrow$ (equal)

Within a tier, work left to right if no parentheses are given. So $p \to \sim q \vee r$ means $p \to ((\sim q) \vee r)$ — but parenthesise explicitly rather than relying on the tie-breaking rule.

Notation, not to be blurred: the biconditional $\leftrightarrow$ is a connective used to **combine** statements (like $+$, $-$, $\times$, $\div$ with numbers). Logical equivalence $\equiv$ is used when **comparing** two statements (like $=$ with numbers). $a \leftrightarrow b$ is a statement; $a \equiv b$ is a claim *about* two statements.

See also

- logical-connectives[3]
- logical-equivalence-laws[4]
- sets-and-set-notation[5] — vacuous truth doing real work, in $\emptyset \subseteq A$
- 2026-07-30-conditional-statements[6] — Lecture 3
- practice-problems[7] — §3 Conditional Statements, with full worked solutions

[3] courses/math1061/logical-connectives.pdf
[4] courses/math1061/logical-equivalence-laws.pdf
[5] courses/math1061/sets-and-set-notation.pdf
[6] courses/math1061/2026-07-30-conditional-statements.pdf
[7] courses/math1061/practice-problems.pdf