Lecture 4 — Valid and Invalid Arguments

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Pre-work: Video 004 (Valid & Invalid Arguments), Video 005 (Methods for Determining Validity). See valid-argument-forms[1] for the reference definitions/table this lecture builds on.

Learning goals

1. Understand the difference between a valid argument and an invalid argument.
2. Use a truth table to determine the validity of an argument.
3. Use rules of inference to demonstrate the validity of an argument.
4. Look for truth values that demonstrate an argument is invalid.

Recall

$$p \to q \ \equiv \sim p \vee q \ \equiv \sim q \to \sim p$$

$\sim p \vee q$ is the **definition** of $\to$, and $\sim q \to \sim p$ is the **contrapositive**. Both are used as justifications in Argument 1(c) below — line 3 is the contrapositive, line 6 the definition of $\to$ — so it is worth having them to hand before starting.

[1] courses/math1061/valid-argument-forms.pdf

Valid vs invalid arguments

An **argument** is a sequence of statements — premises and a conclusion. An argument is **valid** if the conjunction of the premises implies the conclusion: whenever all of the premises are true, the conclusion is also true. Stated as one formula, the argument is valid exactly when

$$(\text{Premise}_1 \wedge \text{Premise}_2 \wedge \cdots \wedge \text{Last Premise}) \rightarrow \text{Conclusion}$$

is a **tautology**.

The first example is the pink shirt argument — "if today is Monday, then I am wearing a pink shirt; today is Monday; therefore I am wearing a pink shirt" — which is **valid even though its first premise is false and its conclusion is false**. Its form is Modus Ponens. It is worked both ways (row check and tautology check) in determining-argument-validity[2].

See valid-argument-forms[3] for the formal definition and the standard forms (Modus Ponens, Modus Tollens, etc.).

Questions about validity

How can you have a valid argument with a false conclusion? The pink shirt argument is one. Validity is a property of the *form*, not of the facts. It says the conclusion follows *if* the premises hold; it never claims they do.

Why is an argument valid if it has a false premise? The premises sit in the antecedent of $(\text{Premise}_1 \wedge \cdots) \rightarrow \text{Conclusion}$. A false premise makes that conjunction false, and a conditional with a false antecedent is true (the two F-antecedent rows of the $p \rightarrow q$ table). So a row containing a false premise is true whatever the conclusion does, and can never break the tautology. It does *not* follow that an argument with a false premise is automatically valid: the rows where every premise is true still have to be checked.

Do we have to memorise the rules of inference? No. The forms table is on the *MATH1061/MATH7861 Examination Formula Page*, so the list is in front of you in the exam and the work left to do is recognising which form an argument matches and naming it in a justification.

[2] courses/math1061/determining-argument-validity.pdf
[3] courses/math1061/valid-argument-forms.pdf

Three arguments, three methods

Three arguments, worked by three different methods. Full solutions for all of them are in determining-argument-validity[4].

Argument 1: Let $w =$ "wages are raised", $b =$ "buying increases", $d =$ "there is a depression".

> If wages are raised, buying increases. If there is a depression, buying does not increase. Therefore there is not a depression or wages are not raised.

- **(a)** Write this argument in symbolic form. $\rightarrow w \rightarrow b,\ \ d \rightarrow \sim b,\ \ \therefore \sim d \vee \sim w.$
- **(b)** Use a truth table to determine whether the argument is valid. $\rightarrow$ **Valid**: build the eight-row table with $\sim w$, $\sim b$, $\sim d$ as helper columns, find the four rows where both premises are T (rows 2, 6, 7, 8), and check the conclusion is T in each.
- **(c)** Use the laws of logical equivalence and rules of inference to show the argument is valid. $\rightarrow$ **Valid**: contrapose premise 2 into $b \rightarrow \sim d$, chain with premise 1 by Transitivity, then unpack $w \rightarrow \sim d$ into the required disjunction.

Argument 2 (Q1 from the Methods video pre-class questions):

$$\begin{array}{ll} 1. & p \rightarrow q \\ 2. & p \vee r \\ 3. & p \vee \sim r \\ \therefore & q \end{array}$$

- **(a)** Use the laws of logical equivalence and rules of inference to show this argument is valid. $\rightarrow$ **Valid**: conjoin the two disjunctive premises, distribute, and p falls out; Modus Ponens finishes it.
- **(b)** Alternatively, try to find truth values that make all the premises true but the conclusion false — if none exist, the argument is valid. $\rightarrow$ None exist: the search forces r to be both true and false.

Argument 3 (Q2 from the Methods video pre-class questions):

$$\begin{array}{ll} 1. & p \rightarrow q \\ 2. & q \rightarrow r \\ 3. & \sim p \vee \sim q \\ \therefore & r \end{array}$$

Try to find truth values that make all the premises true but the conclusion false. $\rightarrow$ They exist: p, q, r all false makes every premise true and the conclusion false, so this argument is **invalid**.

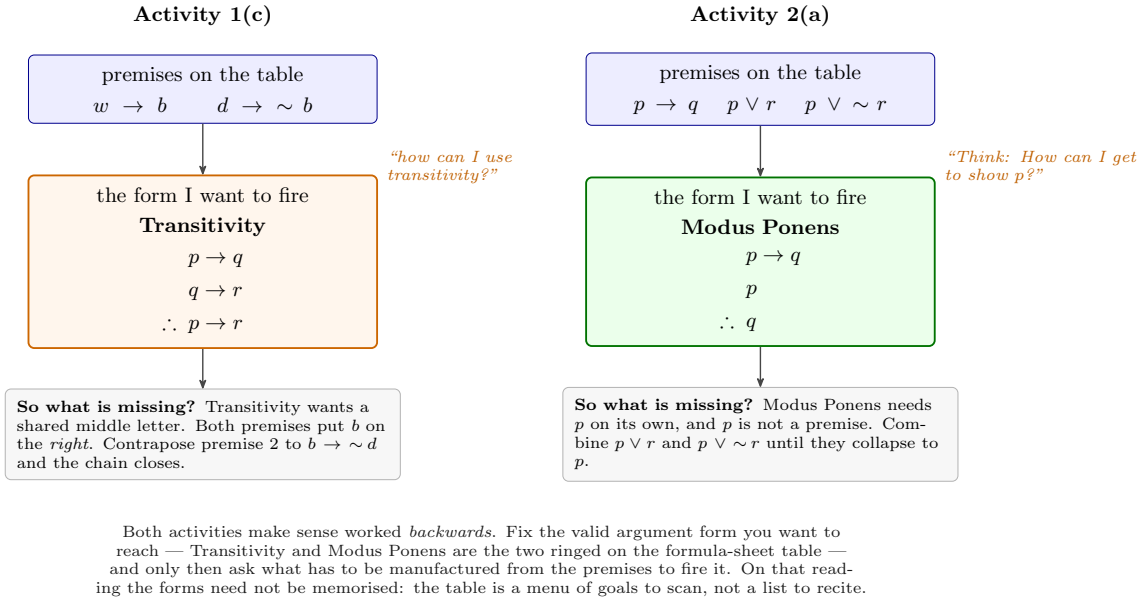
[4] courses/math1061/determining-argument-validity.pdf

Extra practice: revisit Argument 1 — run the same invalidity search on it. The conclusion being false forces d and w both true, premise 1 then forces b true, and premise 2 is now false. Contradiction, so Argument 1 is valid a third time.

Driving a rules-of-inference proof

Each of the two rules-of-inference proofs is steered towards one form: Transitivity for Argument 1(c), Modus Ponens for Argument 2(a). They land in different places — Transitivity is the pivot of Argument 1(c), at line 5 of 7, while Modus Ponens is the last line of Argument 2(a). The question to ask of each argument is how to reach that form: "how can I use transitivity?" for Argument 1(c), and "how can I get to show p?" for Argument 2(a).

The habit is to pick the valid argument form you want to reach, *then* work out what has to be manufactured from the premises to fire it. It is the transferable part, and both proofs are consistent with it.



Both activities make sense worked *backwards*. Fix the valid argument form you want to reach — Transitivity and Modus Ponens are the two ringed on the formula-sheet table — and only then ask what has to be manufactured from the premises to fire it. On that reading the forms need not be memorised: the table is a menu of goals to scan, not a list to recite.

Figure 1: The two rules-of-inference proofs, Argument 1(c) and Argument 2(a), arranged around the form each one is steered towards: Transitivity (orange border) and Modus Ponens (green border). Each panel goes from the premises to the form to reach, and then to what is still missing to fire it. Worked backwards like this, the forms table is a menu of goals to scan, not a list to recite.

Every derived line needs a written justification. Naming the law or form applied, and the lines it came from ("from by contrapositive"; "from & by transitivity"), is part of the answer, not a courtesy.

Which method to reach for

	truth table	rules of inference	invalidity search
Used here on	the pink shirt example, Argument 1	Arguments 1, 2	Arguments 1, 2, 3
Can prove valid	yes	yes	yes
Can prove invalid	yes	**no**	yes
Effort with n variables	2^n rows	varies; may stall	usually a handful of forced values
Gives you the counterexample	you have to spot the row	never	yes, directly

The invalidity search is the only method used on all three arguments, and Argument 3 — the one invalid argument of the three — can only be settled by it or by a full truth table. A stalled rules-of-inference attempt proves nothing at all.

See also

- valid-argument-forms[5] — the named argument forms (Modus Ponens, Modus Tollens, Elimination, Generalization, Specialization, as the formula page spells them)
- determining-argument-validity[6] — the truth-table, rules-of-inference and invalidity methods worked through on all three arguments
- logical-equivalence-laws[7] — the laws the inference proofs lean on
- 2026-07-30-conditional-statements[8] — Lecture 3 · 2026-08-06-quantified-statements[9] — Lecture 5
- practice-problems[10] — §4 Valid and Invalid Arguments, with full worked solutions
- practice-problems[11] — §5 Methods for Determining Validity, with full worked solutions

[5] courses/math1061/valid-argument-forms.pdf
[6] courses/math1061/determining-argument-validity.pdf
[7] courses/math1061/logical-equivalence-laws.pdf
[8] courses/math1061/2026-07-30-conditional-statements.pdf
[9] courses/math1061/2026-08-06-quantified-statements.pdf
[10] courses/math1061/practice-problems.pdf
[11] courses/math1061/practice-problems.pdf