Lecture 3 — Conditional Statements

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Pre-work: Video 003 (Conditional Statements). See conditional-statements[1] for the reference definitions this lecture builds on. **Note**: Applied Classes start in Week 2, and they include an assessed component.

The proofs of $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \equiv p$ — one by truth table and two by chains of laws — are at the end of Lecture 2[2].

Learning goals

1. Understand the connectives *if…then* ($\rightarrow$) and *if and only if* ($\leftrightarrow$).
2. Be familiar with common logical equivalences involving *if…then*:

$$p \rightarrow q \equiv \sim p \vee q \equiv \sim q \rightarrow \sim p$$

   Know this — the two rewrites are the whole toolkit for everything below.
3. Be able to negate a conditional statement.
4. Understand the meaning of necessary and sufficient conditions.

[1] courses/math1061/conditional-statements.pdf
[2] courses/math1061/2026-07-30-logical-form-and-equivalence.pdf

Common questions

- **How can $p \to q$ be true when p is false?** Because that is how it is defined. The bottom two rows of the table are a definition, not a claim about the world.
- **Can we use any laws for "p implies q"?** Replace $p \to q$ with $\sim p \vee q$ and then use the laws. The laws of logical equivalence are stated only for $\sim, \wedge, \vee$, so this substitution is what gives you access to them.
- **For necessary and sufficient conditions, which one signifies IF, and which ONLY IF?** "If" is the *sufficient* half, "only if" is the *necessary* half.

Conditional ($\to$)

p	q	$p \to q$
T	T	T
T	F	**F**
F	T	T
F	F	T

Exactly one row makes it false: **true hypothesis, false conclusion**. See the containment figure in conditional-statements[3]: the situations making p true sit **inside** the situations making q true, which is why nothing outside p can falsify the conditional.

By definition — this is also called the **disjunctive form** of $\to$ —

$$p \to q \equiv \sim p \vee q$$

Contrapositive, converse, inverse

Name	Form	
Conditional	$p \to q$	**logically equivalent** to each other
Contrapositive	$\sim q \to \sim p$	
Converse	$q \to p$	**logically equivalent** to each other
Inverse	$\sim p \to \sim q$	

Reminder: $p \to q \not\equiv q \to p$. The four forms and the only equivalences among them are laid out as a figure in conditional-statements[4].

[3] courses/math1061/conditional-statements.pdf
[4] courses/math1061/conditional-statements.pdf

English phrasings

$p \to q$ can be expressed in many ways in English. The following are logically equivalent:

- p implies q
- If p then q
- q **if** p
- p is **sufficient** for q
- p **only if** q
- q is **necessary** for p

Watch where the letters land: "q if p" puts the hypothesis *second*, and "p only if q" puts the hypothesis *first*. Those two lines are the ones worth memorising, because they are the ones the necessary/sufficient vocabulary is built on.

Biconditional

$$p \leftrightarrow q \;\equiv\; (p \to q) \wedge (q \to p)$$

Worked: equivalents and negation of a conditional

(a) Which of the following statements have the same meaning as: "If the file is missing, then the program crashes"? (Multiple answers allowed.)

Let p = "the file is missing" and q = "the program crashes", so the statement is $p \to q$.

Answer: B, C and E.

(b) Write the NEGATION of the original statement.

$$\sim (p \to q) \;\equiv \sim (\sim p \vee q) \;\equiv\; p \wedge \sim q$$

by the definition of $\to$, then De Morgan's law and the double negative law. In English:

> **The file is missing but the program didn't crash.**

The negation of a conditional is an **and**, never another conditional.

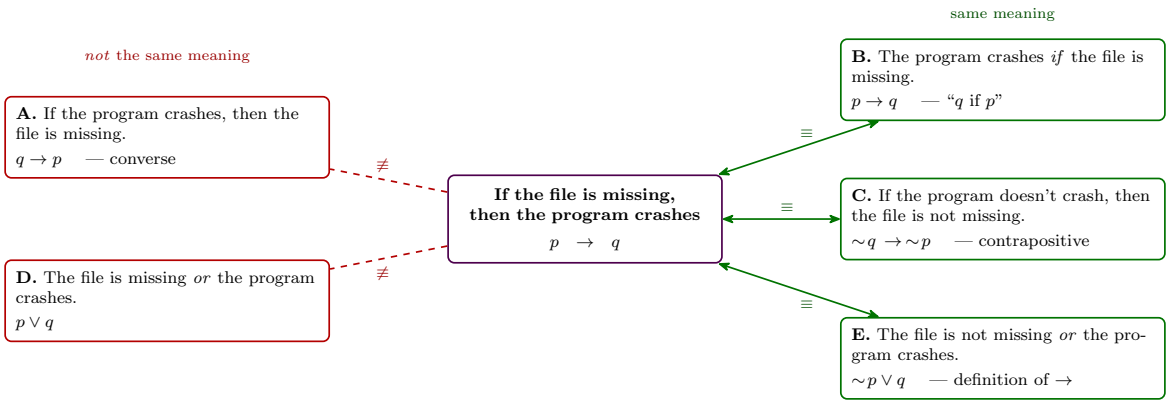

Figure 1: The five options and their verdicts, each translated into symbols: green has the same meaning as the original, red does not. The verdict follows from the symbols alone: B is the same conditional with the 'if' clause second, C is its contrapositive, E is its disjunctive form. A is the converse and D is a plain disjunction — neither is equivalent. Translate before judging: A and B are built from the same two clauses in the same order — all that moves is which clause the 'if' attaches to, and that alone reverses the conditional.

Worked: the inverse is not the negation

Demonstrate that $\sim (p \to q) \not\equiv \sim p \to \sim q$**.**

> Let p be true and q be true. Then $(p \to q)$ is true, so $\sim (p \to q)$ is **false**. BUT $\sim p \to \sim q$ is **true**. $\therefore \sim (p \to q) \not\equiv \sim p \to \sim q$.

The third line holds because $\sim p \to \sim q$ evaluates as F $\to$ F, true by the bottom row of the table. This is the shape for demonstrating $\not\equiv$: name one assignment, evaluate both sides on it, then state the conclusion. No truth table required.

$\sim p \to \sim q$ is the **inverse** — offering it as the negation of $p \to q$ is the classic error that learning goal 3 is aimed at.

Order of operations (extended)

1. $\sim$
2. $\wedge, \vee$ (equal) — use parentheses to specify; if none, work from left to right
3. $\to, \leftrightarrow$ (equal) — use parentheses to specify; if none, work from left to right

So the statement form $p \to \sim q \vee r$ means $p \to ((\sim q) \vee r)$, **but parentheses should always be used to avoid confusion.**

Note on notation. The biconditional $\leftrightarrow$ is a connective, used to **combine** statements (similar to $+, -, \times, \div$ … with numbers). Logical equivalence $\equiv$ is used when **comparing** two statements (similar to $=$ with numbers). Do not blur the two: $a \leftrightarrow b$ is itself a statement, $a \equiv b$ is a claim about two statements.

Worked: an equivalence proved two ways

Let p, q and r be statements. Show that $p \to (q \vee r) \;\equiv\; (p \wedge \sim q) \to r$.

There are two solutions.

Solution 1 — one chain, all the way across:

$$\begin{aligned} p \to (q \vee r) &\equiv \sim p \vee (q \vee r) && \text{by definition of } \to \\ &\equiv (\sim p \vee q) \vee r && \text{by the associative law} \\ &\equiv (\sim p \vee \sim\sim q) \vee r && \text{by the double negative law} \\ &\equiv \sim (p \wedge \sim q) \vee r && \text{by De Morgan's law} \\ &\equiv (p \wedge \sim q) \to r && \text{by definition of } \to \end{aligned}$$

Solution 2 — work both sides down to the same thing:

$$\begin{aligned} \text{LHS:}\quad p \to (q \vee r) &\equiv \sim p \vee (q \vee r) && \text{definition of } \to \\ &\equiv (\sim p \vee q) \vee r && \text{associative} \\ \text{RHS:}\quad (p \wedge \sim q) \to r &\equiv \sim (p \wedge \sim q) \vee r && \text{definition of } \to \\ &\equiv (\sim p \vee q) \vee r && \text{De Morgan's and double negative} \end{aligned}$$

Both sides reach $(\sim p \vee q) \vee r$ — **the same** — so the two forms are equivalent.

Solution 2 avoids having to guess which direction to push a negation — each side is simplified independently until they meet. Both are complete proofs.

Necessary and sufficient conditions

> p is a **necessary condition** for q means "if $\sim p$ then $\sim q$", or equivalently "if q then p" $(q \to p)$, or "q **only if** p".
>
> p is a **sufficient condition** for q means "if p then q", or equivalently "q **if** p".

The two phrasings of *necessary* are contrapositives of each other, which is why they agree.

Worked — necessary but not sufficient. *Being a non-negative integer is necessary for a number to be the square of an integer, but it is not sufficient.*

- If a number is the square of an integer, then it is non-negative — so -9 **cannot** be the square of an integer.
- But 7 is a non-negative integer and is **not** the square of an integer, so the condition does not suffice.

Worked — sufficient but not necessary. *Being an integer that is divisible by 4 is sufficient for being an even integer, but it is not necessary.*

- 8 is divisible by 4, so we know that 8 is even — no further checking needed.
- But 10 is an even integer and is **not** divisible by 4, so the condition is not required.

Biconditional as necessary *and* sufficient

The three rows below say the same thing, and each splits at the same seam:

Whole statement		the **necessary** half		the **sufficient** half
$a \leftrightarrow b$	$\equiv$	$(a \to b)$	$\wedge$	$(b \to a)$
a if and only if b	$\equiv$	a **only if** b	and	a **if** b
b is necessary and sufficient for a	$\equiv$	b is necessary for a	and	b is sufficient for a

Read down the third column and you get three names for one half of the biconditional; down the fifth, three names for the other. That answers the earlier question: **"only if" marks the necessary half, "if" marks the sufficient half.** The same table is a colour-aligned figure in conditional-statements[5].

Worked: conditions for divisibility by 6

Let n be a positive integer.

[5] courses/math1061/conditional-statements.pdf

	Asked	Answer	Why
(a)	A condition **necessary but not sufficient** for n to be divisible by 6	n **is divisible by 3**	Every multiple of 6 is a multiple of 3, so it is necessary; $n = 3$ is divisible by 3 but not by 6, so it is not sufficient
(b)	A condition **sufficient but not necessary** for n to be divisible by 6	$n = 12$	12 is divisible by 6, so it suffices; $n = 6$ is divisible by 6 without being 12, so it is not necessary
(c)	A **necessary and sufficient** condition for n to be divisible by 6	n **is divisible by both 2 and 3**	Both directions hold, so this is a genuine biconditional

A *sufficient but not necessary* condition can be cheap — pinning n to a single value is sufficient for anything true of that value.

See also

- conditional-statements[6] — the reference definitions, the four forms, and the necessary/sufficient vocabulary
- logical-equivalence-laws[7] — the laws used in the negation and the two-way proof above
- 2026-07-30-logical-form-and-equivalence[8] — Lecture 2 · 2026-08-03-valid-and-invalid-arguments[9] — Lecture 4
- practice-problems[10] — §3 Conditional Statements, with full worked solutions

[6] courses/math1061/conditional-statements.pdf
[7] courses/math1061/logical-equivalence-laws.pdf
[8] courses/math1061/2026-07-30-logical-form-and-equivalence.pdf
[9] courses/math1061/2026-08-03-valid-and-invalid-arguments.pdf
[10] courses/math1061/practice-problems.pdf